CONSTRUCTING THEORIES:

A CASE STUDY IN GEOMETRY WITH AN EXCURSION INTO BIOLOGY

K P Mohanan and Tara Mohanan

Constructing Theories:

A Case Study in Geometry with an excursion into Biology

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This book is an attempt to introduce the reader to the art and craft of the construction (and evaluation) of theories. While we use geometry as the terrain for practice, our focus is on the methodological strategies and techniques of value in all domains of theory construction, not on the 'facts' of geometry that students learn in textbooks.

We expect that the units in the book will be useful for researchers and research students across disciplines, but have also tried to make the content and the style accessible to high school students.

NOTE: If you find any errors in these pages, please alert us at:
tara.mohanan@gmail.com and mohanan.kp@gmail.com

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UNIT 1: A THEORY OF TRIANGLES

1.1 Right-Angled Triangles

Here are a few examples of Right-Angled Triangles.

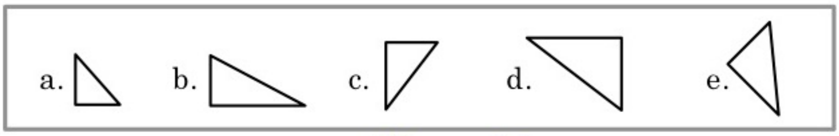

Figure 1-1

What are the properties of Right Angled Triangles? Let us make a list:

1) Properties of Right-Angled Triangles (RATs)

a) An RAT has three angles.
b) An RAT has three straight lines.
c) One of the angles of an RAT is a right angle.
d) The area of an RAT is equal to half the product of the lengths of the two sides adjacent to the right angle.
e) The length of the square of the side opposite the right angle is equal to the sum of the squares of the lengths of the other two sides.

1.2 Equilateral Triangles

Here are a few examples Equilateral Triangles:

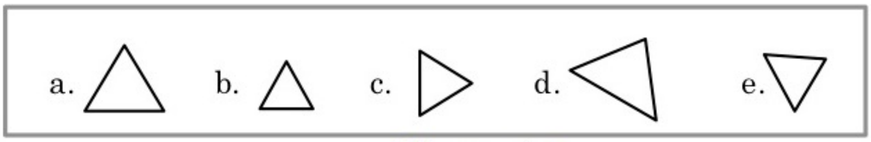

Figure 1-2

Once again, let us make a list, this time of the properties of ETs:

2) Properties of ETs:

a) An ET has three angles.
b) An ET has three straight lines.
c) In an ET, the size of any angle is equal to the size of any other angle.
d) The area of an ET is equal to half the product of the lengths of any one of the sides and the length of the perpendicular from the opposite angle to that side.
e) In an ET, the length of any one side is equal to the length of any other side.

At this point, you might experience a mild discomfort. In (2a-b), we are repeating what we said about RATs in (1a-b) as properties of ETs. While we agree that these statements are true, do we need to repeat them? How do we avoid this unnecessary repetition?

1.3 Triangles

One solution is to begin by noting that RATs and ETs are both Triangles. They (RATs and ETs) are subcategories of the category 'Triangle'. Examples of triangles include not only those in Figs. 1-1 and 1-2, but also those in Fig. 1-3:

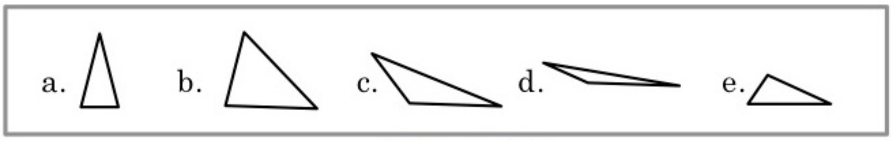

Figure 1-3

What about the properties of Triangles? Here they are:

3) Properties of Triangles

a) A Triangle has three angles.

b) A Triangle has three straight lines.

c) In a Triangle, the sum of the angles is equal to twice the sum of a right angle.

d) The area of a Triangle is equal to half the product of the length of any one of the sides, and the length of the perpendicular from the opposite angle to that side.

e) In a Triangle, the length of a side increases as the length of the perpendicular to that side from the opposite angle increases, provided the angles remain unchanged.

We now see that (1a-b) and (2a-b) follow from (3a-b). So given (3a-b), repeating the statements as (1a-b) and (2a-b) is unnecessary, or ***redundant***.

1.4 Derivation

How exactly do the statements in (1a-b) and (2a-b) follow from (3a-b)? What is the general principle that allows us to derive (1a-b) and (2a-b) from (3a-b)?

We had noted earlier that:

4) The categories of RATs and ETs are sub-categories of Triangles.

Given (4), we can derive (1a-b) and (2a-b) from (3a-b) if we postulate the following general principle:

5) **General Principle**:
The properties of a category are inherited by their subcategories.

The derivation given below illustrates the application of this general principle.

6) **Derivation**

A Triangle has three angles. (3a)

The categories of RATs and ETs are sub-categories of Triangles. (4)

The properties of a category are inherited by their subcategories. (5)

Hence, an RAT and an ET have three angles. (1a), (2a)

Exercise 1

Derive (1d) and (2d) from (3d).

1.5 The Structure of Theories

Derivations like the one illustrated above, and the logical structure that allows for such derivations, are the hallmarks of theories, distinguishing them from mere descriptions. As far as we know, Euclid in Ancient Greece and Panini in Ancient India were the people who originally developed theories in the sense of the term 'theory' in mathematical and scientific inquiry. Panini's work, called *Ashtadhyahyi*, is not accessible to students, but the English translation of Euclid's Elements is accessible to those who are willing to struggle with it. It is downloadable at:

http://farside.ph.utexas.edu/Books/Euclid/Elements.pdf

but only after completing *Constructing Theories of Geometry*.

1.5.1 Derivations

A derivation is a sequence of steps of reasoning. What makes derivations possible are a set of assumptions that we take as the premises of an argument. Given these premises, the derivation yields a conclusion.

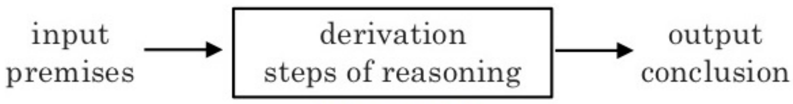

Figure 1-4

1.5.2 Definitions, Postulates, and Common Notions

Euclid distinguishes three kinds of premises: definitions, postulates, and common notions.

The ***definition*** of a concept is a statement that tells us what comes under that concept and what doesn't. For example, a definition of even numbers tells us which integers are even and which ones are not. A definition of mammals tells us which organisms are mammals and which ones are not. A definition of democracy tells us which systems of decision making are democratic and which ones are not.

Here are some examples of definitions in geometry:

Right Angle (DEF): An angle is a Right Angle if and only if it is formed by two lines that are perpendicular to each other.

Obtuse Angle (DEF): An angle is an Obtuse Angle if and only if it is formed greater than a right-angle.

Acute Angle (DEF): An angle is an Acute Angle if and only if it is less than a right-angle.

What, then, is the definition of 'perpendicular', that appears in the definition of a Right Angle?

Perpendicular (DEF): Line A is perpendicular to line B if and only if, when A stands on B and the adjacent angles are equal.

Euclid used the term ***postulate*** to denote premises that are specific to geometry. Given below are examples of postulates:

Postulate 1: Given any two distinct points, one and only one straight-line exists between them.

Postulate 2: Any finite straight-line can be extended indefinitely as a straight line.

The term ***common notion*** in Euclid denotes premises that apply to mathematics in general, not just to geometry. Here are examples of common notions:

Common Notion 1: Things that are equal to the same thing are also equal to one another.

Common Notion 2: If equal things are added to equal things, then the wholes are equal.

We will adopt the essence of the Euclidean distinction between postulates and common notions. But we will call them ***axioms***, and distinguish two categories of axioms: ***general axioms*** and ***discipline-specific axioms***.

1.5.3 Axioms and Axiomatic Systems

We will use the term ***General Axioms*** to denote those axioms that are relevant in all disciplines and discipline groups — whether geometry, number theory, mathematics, astronomy, physics, the physical sciences, evolutionary biology, developmental biology, organismic biology, cognitive science, linguistics, sociology, anthropology, economics, history, or philosophy. The General Principle in (5) is an example of such a General Axiom. Whether the common notions that Euclid set up fall under General Axioms that hold on all domains of research, or are specific to a particular kind of research, is an issue that we will pick up later.

Euclid saw axioms as 'self-evident truths'. However, subsequent develop-ments in mathematics show that they are ***assumptions***. This means that they are not self-evident truths, but premises that are set up as starting points for particular theories.

A system of knowledge that exhibits the property of deriving conclusions from axioms and definitions is called an ***axiomatic system***. Any fully fleshed-out explicit theory, whether in mathematics, the physical-biological-human sciences, or the humanities includes an axiomatic system.

Exercise 2

a. Try to group the statements in (1) (2) and (3) into premises and conclusions in such a way that the number of premises is minimised. Derive (1d) from (3d)
b. Make sure that you have a derivation for each conclusion.
c. Separate the premises into definitions and axioms.
d. Separate the axioms into axioms into General Axioms and Discipline-Specific Axioms.

Note: The term 'discipline-specific' in (d) is a bit murky because we do not have a clear idea of what a discipline is and what a group of disciplines is. E.g. is molecular biology a discipline or a field within a discipline called life sciences? Is zoology a discipline or a field with life sciences? Do the study of society (covering human sociology and non-human sociology form a single discipline? If they are two disciplines, do they constitute a single discipline group?) Granted this murkiness, it still useful to invest some time on (d).

1.6 Theories vs. Descriptions

The statements illustrated in (1), (2) and (3) are descriptions, not theories. A description is a collection of statements about a given entity. Thus, the statements in (7) are part of a description of one of the members of a human being called Zeno:

7)
a. Zeno is a human being.
b. Zeno is an adult.
c. Zeno is a living organism.
d. Zeno has eukaryotic cells.
e. Zeno has two eyes.
f. Zeno has a mouth.
g. Zeno has a heart.
h. Zeno has vertebrae.
j. Zeno has bones.
k. Zeno is male.
l. Zeno is 180 cms tall.
m. Zeno weighs 82 kilograms.
n. Zeno is diabetic.
o. Zeno is prone to depression.
p. Zeno works as a banker.
q. Zeno has two siblings.
r. Zeno is married to Athena.
s. Zeno has three children.

And the statements in (8) are part of a description of the human species:

8)
a. Humans are living organisms.
b. Humans have eukaryotic cells.
c. Humans have two eyes.
d. Humans have a mouth.
e. Humans have a heart.
f. Humans have vertebrae.
g. Humans have bones.

An important characteristic of a theory is that it separates what is predictable from what is idiosyncratic. By 'predictable', we mean something that can be inferred from some other information. Thus, we can predict that Zeno has eukaryotic cells (7d) from the statements that Zeno is a human being (7a) and humans have eukaryotic cells (8b). A theory separates these two kinds of statements by deriving what is predictable from the premises (axioms and definitions) of the theory.

Exercise 3

a. Do a google search for butterflies, ants, and insects, and for each category, write a set of sentences to describe the anatomy, along the lines illustrated in (1), (2) and (3). Do not go beyond what you judge to be ten most important points. Otherwise, this exercise will be way too time consuming.
b. Try to group the statements for each category into premises and conclusions in such a way that the number of premises is minimised.
c. Make sure that you have a derivation for each conclusion.
d. Separate the premises into definitions and axioms.
e. Separate the axioms into General Axioms and Discipline-Specific Axioms.

In the Units that follow, we will learn how to construct and evaluate theories by separating what is predictable from what is not predictable, and deducing what is predictable (conclusions) from what is not predictable (premises).

While learning the rudiments of this methodological strategy, we will also learn a number of other strategies and techniques of theory construction, and be introduced to some of the norms governing theory construction, and knowledge construction in general.

UNIT 2: A THEORY OF QUADRILATERALS

2.1 Squares

Here are a few examples of Squares.

Figure 2-1

What are the properties of Squares? Let us make a list:

1) Properties of Squares

a) A Square has four angles.

b) A Square has four straight lines.

c) The size of any angle in a Square is equal to that of every other angle in that Square.

d) The length of each line is equal to that of every other line in that Square.

e) In a Square, the sum of the sizes of angles is equal to four times a right angle.

f) The area of a Square is equal to the product of the lengths of any two of the sides of that Square.

g) The square of the length of the diagonal of a Square is equal to twice the sum of the square of the length of any of its sides.

Exercise 1

a) Based on your current knowledge of geometry, add to the list in (1).

b) Do an Internet search to expand your list even further. For instance, try these:

"Properties of a rhombus, a rectangle, and a square"
(https://www.youtube.com/watch?v=3i2yp-lI_V4)

"Theorems dealing with rectangles, rhombuses, and squares"
(https://mathbitsnotebook.com/Geometry/Quadrilaterals/QDRectangle.html)

c) Based on your knowledge of geometry, make a list of the properties of Rectangles, Rhombuses, and Parallelograms. Drawing on the way we put together the properties of RATs, ETs and Triangles, try to put together the properties of Squares, Rectangles, Rhombuses and Parallelograms. Having done that, set up statements of subcategories among them in such a way that using the General Axiom (5) in Unit 1, you can deduce as many properties of these geometrical objects as possible from a small number of premises.

2.2 Triangles and Quadrilaterals

Chances are that the exercises in §2.1 led you to propose that Squares, Rectangles, Rhombuses and Parallelograms are subcategories of Quadrilaterals.

Did you, by any chance, also think of proposing that Squares are subcategories of Rectangles? That Rectangles are subcategories of Parallelograms? That Squares are also subcategories of Rhombuses? If you didn't happen to think of these possibilities, reflect on them now and write down your subcategorization statements.

Having explored the ***properties*** of both Triangles and Quadrilaterals, we can now look at the ***relations*** between them. For instance, is it possible to divide a Square into two Triangles? If it is, what kind of Triangle do you get? RATs? ETs? Isosceles Triangles?

Is it possible to put together one or more Triangles into a Square? Any category of Triangles?

Explore these relations, and make a list of the statements that express them. Then try to derive the relations from the premises you have already come up with.

Now consider Pentagons. Do you see a set of shared properties in the categories of Triangles, Squares, and Pentagons? Try to formulate statements that describe what they have in common.

In (1), we stated a number of properties of Squares. Suppose we delete one or more of these properties in (a)-(d). Would you then get figures which are not Squares? What would their categories be? Spend a few minutes reflecting on this question.

2.3 Triangles, Quadrilaterals, Pentagons, Hexagons …

It is time to write descriptions of the categories of geometric figures that we have been investigating.

In Unit 1, we saw a number of repetitions in the descriptions of RATs, ETs, and Triangles. To eliminate these repetitions, we did the following:

- ~ treat RATs and ETs as subcategories of the category of Triangles;
- ~ set up a General Principle, like (5), on the subcategories of a category;
- ~ remove the redundant statements in the descriptions of RATs and ETs; and
- ~ derive those predictable statements from the premises.

Can you try to use the same methodological strategy on the descriptions of the categories of geometric figures that you have been investigating, and integrate all of them into a single theory?

Exercise 2

a) Do a Internet search for crows, vultures, and birds. For each category, write a set of sentences to describe the anatomy, along the lines illustrated in (1), (4) and (6). Do not go beyond what you judge to be ten most important points. Otherwise, this exercise will be way too time consuming.

b) Try to group the statements for each category into premises and conclusions in such a way that the number of premises is minimised. the premises are the smallest possible in number.

c) Make sure that you have a derivation for each conclusion.

d) Separate the premises into definitions and axioms.

e) Separate the axioms into General Axioms and Discipline-Specific Axioms.

Exercise 3

a) Do a google search for the vertebra of crows, vultures, and birds. For each category of vertebra, write a set of sentences to describe the structure of the vertebra. Do not go beyond what you judge to be ten most important points. Otherwise, this exercise will be way too time consuming.

b) Try to group the statements for each category of vertebra into premises and conclusions in such a way that the number of premises is minimised.

c) Make sure that you have a derivation for each conclusion.

d) Separate the premises into definitions and axioms.

e) Separate the axioms into axioms into General Axioms and Discipline Specific Axioms.

Exercise 4

a) Do a google search for the vertebra of mice, bats, and mammals. For each category of vertebra, write a set of sentences to describe the structure of the vertebra. Do not go beyond what you judge to be ten most important points. Otherwise, this exercise will be way too time consuming.

b) Try to group the statements for each category of vertebra into premises and conclusions in such a way that the number of premises is minimised.

c) Make sure that you have a derivation for each conclusion.

d) Separate the premises into definitions and axioms.

e) Separate the axioms into General Axioms and Discipline Specific Axioms.

Exercise 5

In Exercise 2 we constructed a theory of the categories of organisms. In Exercises 3 and 4, in contrast, we constructed a theory of the categories of a particular organ (vertebra) in these categories of organisms. Think of a way to construct a single theory that includes categories of both organisms and categories of organs.

Exercise 6

In Aristotle's categorisation of animate entities, the first division was human vs non-human. The category of non-human was further categorised as animal vs. plant. In the classification proposed by Carl Linnaeus, on the other hand, the first division was animal vs. plant. Humans came under animals. A third possible categorisation would be human, animal, and plant.

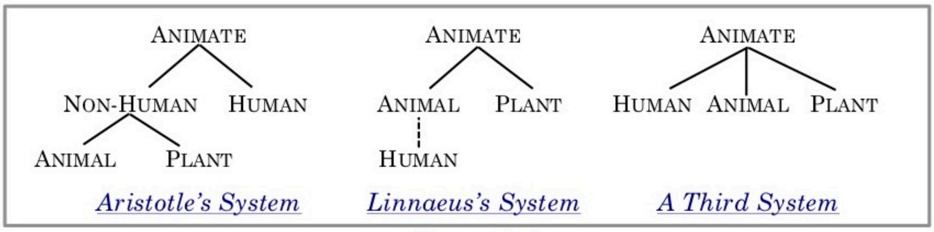

Figure 2-2

For a theory of animate entities that maximises predictions and minimises premises, which of these systems would you choose? State your reasons, drawing on what you have learnt so far on the strategies and norms of theory construction-and-evaluation.

2.4 Concluding Remarks

In Unit 1, we were introduced to the how the art and craft of theory construction by separating what is predictable from what is not predictable in descriptions, deducing what is predictable from what is not predictable. This methodological strategy is accompanied by the categorisation of the premises as definitions, and the categorisation of axioms into General Axioms and Discipline-Specific Axioms.

In Unit 2, we extended what we leant in Unit 1 to construct at theory that integrates theories of Parallelograms, Rhombuses, Rectangles and Squares. We also practiced the methodology of theory construction to construct a theory of insects, a theory of birds, and a theory of vertebra.

Our decisions on categorisation, as well as on what should be treated as premises and what should be treated as predictions, are based on the norm of

minimising premises and maximising predictions.

When we are faced with the choice between competing theories (including competing categorisations and competing premise-prediction pairings), we choose that option which minimises premises and maximising predictions.

In Unit 3, we will integrate theories of Triangles, Quadrilaterals and Pentagons to construct a theory of Polygons, also covering other Polygons like Hexagons, Septagons, and Octagons. We will then proceed to Lines and Points in Unit 4.

UNIT 3: A THEORY OF POLYGONS

3.1 Looking Back

In Units 1 and 2, we drew attention to some of the ***methodological strategies*** of theory construction:

> ***defining, categorising, abstracting, integrating,*** and ***proving***.

These are relevant in all domains of academic research. You could think of these strategies also as *techniques*, or *tools* used in builiding knowledge. The words we use for these strategies are in their verb form. Corresponding to them are also their noun forms, which we think of as ***concepts*** of academic knowledge and inquiry/research:

> ***definitions, categories, abstractions, integration***, and ***proofs***.

In Unit 2, we unpacked the concept of proof in terms of a structure that involves the concepts of ***premises, derivation,*** and ***conclusion*** (PDC structure of proofs). We made a distinction between two kinds of premises: ***axioms*** and ***definitions***. We saw the methodological strategy of ***axiomatising*** — of explicitly stating the axioms and definitions of a proof, such that we can critically evaluate the ***validity*** of proofs and the ***credibility*** of theories. The concept corresponding to axiomatising, we said, is that of an ***axiomatic system***.

These strategies and concepts were introduced in the specific context of geometry (and biology). However, they play an important role in theory construction in all domains of research, ranging from mathematics, the physical-biological-human sciences, and the humanities. We will therefore refer to them as ***transdisciplinary*** strategies and concepts, *trans-* in the sense that they cut across specialised fields, disciplines, and discipline groups, and exist at a more abstract level. To take an example of what we mean by transdisciplinary, the so-called *theory of classical mechanics* is specific to the highly specialised field that studies gravity and motion within the discipline called physics; but the concept of ***theory*** is a trans-disciplinary concept. It is in this sense that we talk about transdisciplinary concepts.

Take another example. The concepts of gravity, velocity, and acceleration are specific to physics, and the concept of skeletal structure, and of cells being composed of biomolecules is specific to biology. But the concepts of ***structure***, ***compositionality***, and very concept of ***concept*** are transdisciplinary.

In §1.5.3, we made a distinction between axioms that are specific to a particular field or discipline, and ***general axioms*** that are part of all academic knowledge. General axioms are transdisciplinary. We may make the same distinction for definitions. The definition of force as *that which causes a change in the velocity of an inanimate entity* is specific to physics. But the definition of force as *that which causes change* is a transdisciplinary definition, extendable to the animate, even human domains.

The study of knowledge includes the nature of knowledge, ways of arriving at it, critically evaluating it, and proving claims to establish them as part of knowledge. This study is called ***epistemology*** in philosophy, and ***cognitive science*** in the sciences. (Cognising means knowing; and *cognise* and *know* both derive from the Indo-European root *gno*-.)

The kind of epistemology discussed in philosophy textbooks and in philosophy classrooms does not cover the epistemology of ***academic knowledge***. It does not engage with questions such as:

> What is the distinction between mathematical proofs and scientific proofs?
>
> What is the distinction between correlational theories and causal theories?
>
> Given a question, how do we decide what kind of reasoning to use?

In this book, what we are interested in is the *epistemology of academic knowledge* that functions as the foundations for research in all domains of academic knowledge.

In what follows, we will take a closer look at the *concept of proof*, and the role of *categorisation and subcategorization* as an important methodological strategy for *integrating special theories into a general theory*.

3.2 The Concept of Proof

In Unit 2, we introduced the concept of proof in terms of a three-part structure: premises, derivation, and conclusion (PDC). Let us go through a few examples to get a firmer grip on this concept.

Conjecture to be proved: Athena was born in 1903.

Premises

P1. If Plato has a beard, then Aristotle dislikes mangoes.
P2. If Aristotle dislikes mangoes, then Athena was born in 1903.
P3. Plato has a beard.

Derivation: Steps

S1:	If Plato has a beard, then Aristotle dislikes mangoes.	(P1)
S2:	Plato has a beard.	(P3)
S3:	Therefore, Aristotle dislikes mangoes.	(by S1, S2)
S4:	If Aristotle dislikes mangoes, then Athena was born in 1903.	(P2)
S5:	Therefore, Athena was born in 1903.	(by S3, P2)

Conclusion

Athena was born in 1903. (QED)

> NOTE: QED is an abbreviation for the Latin expression Quod Erat Demonstrandum, which means 'that which is to be demonstrated (= to be proved.)

Let us take another example:

Conjecture to be proved: Athena is taller than Apollo.

Premises

P1. Athena is taller than Xena.
P2. Xena is taller than Plato.
P3. Plato is taller than Apollo.

Derivation: Steps

S1:	Athena is taller than Xena.	(P1)
S2:	Xena is taller than Plato.	(P2)
S3:	Therefore, Athena is taller than Plato.	(by S1, S2)
S4:	Plato is taller than Apollo.	(P3)
S5:	Athena is taller than Plato.	(S3)
S6:	Therefore, Athena is taller than Apollo.	(by S4, S5)

Conclusion

Athena is taller than Apollo. (QED)

In Unit 1 (6) [read as: Unit 1, Item (6)), we gave an example of a derivation of the conclusion that Right-Angled Triangles have three angles. Given below is roughly the same proof, but stated in terms of premises, derivation, and conclusion.

Conjecture to be proved: RATs have three angles.

Premises

P1. A Triangle has three angles.
P2. An RAT is subcategory of Triangle.
P3. The properties of a category are inherited by their subcategories.

Derivation: Steps

S1: A Triangle has three angles. (P1)
S2: An RAT is subcategory of Triangles. (P2)
S3: The properties of a category are inherited by its subcategories. (P3)
S4: Therefore, RATs have three angles. (by S1-3)

Conclusion

RATs have three angles. (QED)

The discipline-specific axiom of subcategorization in (P2) and the transdisciplinary axiom of the logical inheritance of properties in (P3) are central to this proof. Proofs that appeal to subcategorization and the accompanying inheritance of properties are essential for integrating the theories of RATs, ETs, and Triangles into a single theory. This point can be generalised as:

> Categorisation and subcategorization serve an important function in the integration of academic knowledge.

We will have more occasions to use this strategy in the integration of other theories.

In school, you must have come across Euclid's Proof of the infinity of prime numbers. If you haven't, please do an Internet search, and take a look at two or three versions of the proof. The following YouTube video is fairly easy to understand:

"Euclid's Proof that there an Infinite Number of Prime Numbers"
at https://www.youtube.com/watch?v=OxGRl0phiB4

3.3 Properties and Relations

In §1.6, we distinguished between ***descriptions*** and ***theories***. A description of an entity is a body of information about that entity. In disciplines like chemistry, the description of an element is a set of structural or behavioural properties of that element. (e.g., copper is malleable but glass is brittle; petrol is inflammable, water is not; and so on.) A theory takes a number of description fragments, generalises them, and then configures the generalisations into a logical structure of premises-derivation-conclusion. A theory is subject to the requirement that the smallest possible number of premises should yield the widest range of conclusions.

When we are looking at the properties of entities that we want our theory to predict (to derive as a conclusion or a theorem), it is useful to think of not only ***properties***, but also ***relations***. Here are a few examples to illustrate the difference:

Property	*Relation*
Zeno IS OLD.	Zeno IS OLDER THAN Plato.
Zeno IS YOUNG.	Zeno IS YOUNGER THAN Plato.
Zeno IS MARRIED.	Zeno IS MARRIED TO Athena.
Zeno IS A TEACHER.	Zeno TEACHES Plato.

As in the case of axioms and definitions, properties and relations can be of a specific field, a discipline, a discipline group, or transdisciplinary. We are interested in both discipline-specific as well as transdisciplinary properties and relations.

Transdisciplinary relations that appear in theories across disciplines include:

Subcategorization: *x* is A SUBCATEGORY OF *y*.

Compositionality: *x* is COMPOSED OF *y, z, …* (Variants: is MADE UP OF / is A CONSTITUENT OF, is DECOMPOSABLE INTO, …)

Ordering: *x* is ORDERED PRIOR TO *y*. (Variants: is RANKED HIGHER THAN; PRECEDES)

Logical consequence: *x* is a LOGICAL CONSEQUENCE OF *y*.

Logical contradiction: *x* LOGICALLY CONTRADICTS *y*.

Equality: *x* IS EQUAL TO *y*. (Variant: IS EQUIVALENT TO, IS AN ANALOGUE OF, IS A HOMOLOGUE OF, …)

Correlation: *x* CORRELATES WITH *y*.

Causation: *x* CAUSES *y*

Instantiation: *x* is AN INSTANCE OF *y*. (Variants: is A MEMBER OF set/category y, is AN EXAMPLE OF y, is A SAMPLE OF y, …)

Negation: x is THE NEGATION OF *y*. (Variant: is THE OPPOSITE OF.) (Also see logical contradiction)

We have already discussed subcategorization (§1.3, §1.4, §2.2, §2.3). In conjunction with the axiom of the logical inheritance of properties (Unit 1 (5)), the use of the subcategory relation allows us deduce otherwise arbitrary properties of a category from its mother category, thereby facilitating the integration of special theories into general theories.

We have seen the appearance of compositionality in geometry in statements like:

A triangle *is composed of* exactly three vertices, and exactly three straight lines connecting them.

A quadrilateral *is composed of* exactly four vertices, and exactly four straight lines connecting them.

A pentagon i*s composed of* exactly five vertices, and exactly five straight lines connecting them.

Compositionality, then, is a ***part-whole relation.*** We say:

A human body is composed of organs;
An organ is composed of tissues;
A tissue is composed of cells;
A cell is composed of molecules;
A molecule is composed of atoms;
An atom is composed of particles.

In the statements of the form "*x is composed of y, z…* ," *x* is the whole, and the rest are parts of that whole.

As in geometry (and in biology), it would be useful to explore compositionality in domains like arithmetic — for instance, we may view the operations of addition, subtraction, multiplication and division as involving compositionality.

The idea of compositionality in arithmetic calls for a short explication. The use of the term operation implies the idea of an input, an action performed on the input (or a process that the input undergoes), and the resultant output:

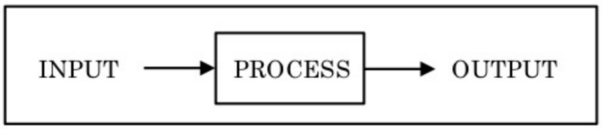

Figure 3.1

This way of thinking of input-process-output based on the metaphor of ACTION and CHANGE is built into the term *operation*. It is also characteristic of the concept of algorithm in computer science, where an algorithm is a set of explicit procedures to do something.

When we say:

10 is the SUM OF 6 and 4, or
10 is a MULTIPLE OF 5,10 is the PRODUCT OF 2 and 5,
10 is DIVISIBLE BY 5,

we may indeed think of these concepts in terms of the input-output metaphor. On the other hand, we may also think of them as binary relations. In this view, "X is the sum of Y+Z" is a binary relation between X on the one hand, and Y and Z on the other. In our example, X is 10, Y is 6 and Z is 4. 10 is divisible by 5 is a relation between 10 and 5. We would like to suggest that the transdisciplinary core of these operations/relations is the concept of compositionality.

It would be useful to pause and reflect on where these relations appear, in all the disciplines that you have encountered in your education/research.

3.4 Towards an Integrated Theory of Polygons

In Units 1 and 2, we introduced the idea of categorising and subcategorising as a methodological tool to integrate two or more special theories into a single general theory. For example, we integrated three otherwise unrelated theories — of RATs, ETs and Triangles — into a single theory of Triangles by postulating that RATs and ETs are subcategories of Triangles (§1.3, §1.4).

Similarly, we integrate the otherwise unrelated theories of Squares, Rectangles, Rhombuses, and Parallelograms into a single theory of Quadrilaterals, by postulating the following axioms:

Squares are a subcategory of Rectangles.
Rectangles are a subcategory of Parallelograms.
Squares are a subcategory of Rhombuses.
Rhombuses are a subcategory of Parallelograms.
Parallelograms are a subcategory of Quadrilaterals.

To this, we may add:

Triangles, Quadrilaterals, Pentagons, Hexagons,…
are subcategories of Polygons.

3.5 Categorization and Compositionality in Conjectures

Let us try some mental exercises. Imagine a Square, by constructing a Square in your mind. In that Square, draw diagonals. How many diagonals can you draw? No more than two, right?

Now imagine a Rectangle, and draw diagonals in it. How many can you draw? Again, no more than two, right? Do the same thought experiment with a Rhombus. How many diagonals? No more than two?

Let us generalise, to a Parallelogram, and to any Quadrilateral.

Based on the sample of pictures in your mind, can you come up with a conjecture on diagonals? To kick off the task, let us begin with one that is maximally general and obviously false, and one that is most specific that we feel is obviously true.

Conjecture 1: A geometric figure can have two but no more than two diagonals.

Conjecture 2: A square can have two but no more than two diagonals.

Exercise 1:

TASK 1: Try to prove (or disprove) Conjectures 1 and 2. To do this, you need to first define the concept of diagonal.

You also need to separate the two strands in the conjecture: showing that:
- (i) there exist geometric figures/squares that can have two diagonals;
- (ii) no geometric figure/square can have more than two diagonals.

TASK 2: Ask yourself if there are geometric figures with three or more diagonals. For this, it would be useful to do a thought experiment of constructing geometric figures with three or more diagonals.

Complete these tasks before you proceed to the next exercise. Otherwise you would miss out on an important opportunity for experiential learning through a minds-on activity (as opposed to a hands-on one).

Exercise 2

TASK 1: Using what you have learnt in this Unit, especially by going through Ex. 1, come up with conjectures on the categories of pentagons, hexagons, and septagons.

Formulate a conjecture on diagonals in polygons.

TASK 2: Now try to formulate a conjecture on diagonals in geometric figures in general. Does this instruction trigger an unease in your mind? Why? Think of a possible reason, and state it in writing, as clearly and precisely as you can.

Let us go back to the geometry that we all learnt before we got to Class 10. We were exposed to the terminology of Right-Angled Triangle, Equilateral Triangle, EquiAngular Triangle, Isosceles Triangle, Square, Rectangle, Parallelogram, Rhombus, Quadrilateral, Pentagon, Hexagon, … Polygon in school. We were also exposed to the terminology of Point, Straight Line, Curved Line, Vertex, Side, and Diagonal. The tasks in Exs. 1 and 2 are meant to trigger in you an experience of what it is like to formulate conjectures, and prove them.

When we do the thought experiment of drawing a diagonal in a square in our mind, we see in our mind's eye, without having to draw it on a piece of paper, that the square is composed of two triangles. Again, without drawing it on paper, we can see that the two triangles are equilateral isosceles triangles.

Let us write down what we have discovered through our thought experiments:

Conjecture 3: Any square can be divided into two right-angled isosceles triangles.

Conjecture 4: Any two right-angled isosceles triangles (of the same size) can be joined to form a square.

Exercise 3

Conjectures 3 and 4 are stated using the idea of a process of dividing or combining: the statements use the metaphor of CHANGE. They use a 'procedural' language.

TASK: Can you restate the conjectures without implying a process of dividing or combining? Such a statement would use the metaphor of an object, and the relation of compositionality. It needs the use of a 'declarative' language.

This translation from procedural language to declarative language is not easy. but it is worth putting effort into, because it will help you train your intuitions, sharpen insight, and develop the capacity to make connections.

The statements that you arrived at in Exs. 1–3 involve the relations between categories, as well as the relation of compositionality. The combination of these two kinds of relations is integral to the concept of ***structure***. This concept is of value not only in geometry, but perhaps in all domains of academic inquiry. In other words, structure as conceptualised above is a transdisciplinary concept. You could view what we are doing in these exercises as beginning to construct ***a theory of the structure*** of polygons.

We hope that you are tickled by — and not confused by — the title, *Structure of Theories* in Unit 1, and the mention of *Theory of Structure* in the preceding paragraph.

Now, the truth of the propositions in the theory you have constructed in Exs. 1–3 may be 'obvious' to you. But we must remind you that it is based on a sample of one square-triangle pair. You cannot do thought experiments on every square-triangle pair in an infinite population of squares and triangles. So, even though they may be 'obvious' to you, you have not yet established them as true propositions. These propositions are conjectures, they are not yet theorems.

There is another reason why these propositions are only *plausible conjectures*, not theorems. Their truth may be obvious to you, but what if they are not obvious to the others in your research community? What if they demand proofs before they accept the propositions as true?

Exercise 4

TASK: Come up with proofs of the conjectures in your theory of squares and triangles. Write down your proofs with as much clarity, precision, and rigour as you can muster.

We could think of the outcome of Ex. 4 as an example of *simulated research*. It is research in the sense that, to do these exercises, you went through the process of inquiry that is needed in research. And it is *simulated* because the theory that you have come up with is already known to the community of mathematicians, so it is not a contribution to the field. (To qualify as research, the outcome of the process must make a contribution to the existing body of knowledge.)

That it is simulated research rather than actual research should not bother you. If you continue along the journey in this book, and practice several times what you have learnt, taking new research questions each time, you will hone your abilities and be ready to make contributions to the existing body of knowledge in your chosen field of study.

Exercise 5

The theory that we have constructed so far is on triangles and squares.

TASK: Generalise it to construct a theory of triangles and quadrilaterals.

Exercise 6

Triangles and quadrilaterals are polygons.

TASK: Construct an integrated theory of polygons, and present it as a written document, with as much clarity, precision, and rigour that you can muster, as an individual or joint project.

Our practice of the methodological strategies for theory construction in Exs. 1–6 have been restricted to the terrain of geometry. For further practice, and for you to get a first-hand experience of seeing what is transdisciplinary about the methodological tools we have used, let us move from geometry to biology.

Exercise 7

TASK: Based on the biology exercises (Exs. 2–5) in Unit 2, construct an integrated theory of the structure of the taxa and of organs discussed in the exercises, and present your theory as a research article.

This is not an exercise that can be done in a few minutes or a few hours. It is a project that could take you a month or two, perhaps even more. You can do it either as an individual project, or a joint project with one or more collaborators.

3.6 Summing up

In Unit 1, we gave a demonstration of how we can construct an integrated theory of triangles, and in Unit 2, showed how we can construct an integrated theory of quadrilaterals. In §3.1–§3.4, we demonstrated how we can construct an integrated theory of polygons.

The transdisciplinary tools and concepts that we have used so far for constructing and evaluating theories in the terrain of geometry include:

structure: categorization and subcategorization, compositionality
generalization, integration
premises (axioms and definitions), derivation and conclusion
conjectures, and theorems/predictions; proof

We have also used these tools and concepts for constructing and evaluating theories in biology, in order to show their transdisciplinary nature.

Unit 4: A Theory of Lines and Points

4.1 Lines without Width and Points without Size

When children first learn about the concept of lines and points, their experiential understanding of these concepts is that of the marks we make on a piece of paper, or on a whiteboard. And more often than not, this is the understanding that they carry into adulthood. This is true, regardless of the words and symbols that accompany the concepts.

Within this conception of lines and points, the marks in Fig. 4-1 are ***lines***, and the ones in Fig. 4-2 are ***points***.

Figure 4-1

Figure 4-2

Such textbook diagrams often get in the way of a conceptual understanding of geometry. The reason is simple: when learners are introduced to Euclidean geometry, they are told:

Lines have no width.
Points have no width and no length; they have zero size.

For learners (whether children or adults), this makes no sense, because the marks in Fig. 4-1 do have width, however small; and the marks in Fig. 4-2 have some area, no matter how small. The experiential concept contradicts what they are taught. But as they have neither the conceptual clarity nor the vocabulary to voice (or even recognise) their discomfort, especially given the culture of obedience and blind faith that education systems promotes, they have to accept what the textbooks and teachers say.

There is an alternative way of introducing learners to the Euclidean concept of lines and points without creating this dissonance. This would be to define lines and points as follows:

LINE (DEF-1): A line is the boundary (edge) of a region.
(DEF-2): A line is a boundary between two regions.

POINT (DEF): A point is an intersection between two lines.

To elaborate, consider Figs. 4-3 and 4-4.

Figure 4-3

Going by DEF-1 of a line, how many LINES does Fig. 4-3 have? Let us take a look through our mind's eye, backed by our physical eye. The lines are:

1) a. the edge of the purple rectangle;
b. the edge of the orange region; and
c. the edge of the green region.

How about if we go by DEF-2? The lines are:

d. the boundary between the purple region and the white region;
e. the boundary between the purple region and the green region;
f. the boundary between the purple region and the orange region; and
g. the boundary between the green region and the orange region.

Do you see a difference in the consequences of DEF-1 and DEF-2 of lines? Notice that the boundary between the green region and the orange region in Fig. 4-3 is a line by DEF-2, but not by DEF-1. This is because, unlike the other lines which are boundaries of regions, and are closed figures, the 'line' between the the green region and the orange region is not a closed figure, nor is it the boundary of any region.

Let us now consider Fig. 4-4.

Figure 4-4

As in the case of Fig. 4-3, the perimeter of the rectangle is the boundary between the purple region of the rectangle and the white region outside it, and is a line by DEF-2.

We also have a white line and an orange line in the figure. Each of them counts as a line by DEF-2, because they form the boundary between two regions of the purple rectangle. They are not lines by DEF-1, however, since they do not form the boundary of a region.

A note to remember: That there are coloured regions in Figs. 4-3 and 4-4 is irrelevant to the concepts we are trying to clarify for ourselves. Colour is not a concept in the world of geometry.

The main point about the above discussion of the concept of line, which we all take for granted as something we know, is to emphasise that

- ~ we cannot talk about a concept in an academic discussion without defining that concept;
- ~ a concept can be defined in more than one ways;
- ~ different definitions may have distinct logical consequences; and
- ~ the definition that we choose as the most appropriate depends on those consequences, and how they contribute to developing a coherent theory.

Given our definitions of a line, are there any lines in Fig. 4-1? Yes, the boundary of the rectangle is a line. But what about the marks inside the rectangle? Given our definition, these marks are not lines, because they do not mark the boundary of a region, nor the boundary between two regions.

Let us turn to points for a moment. Given our definition of a point as an intersection between two lines, the intersection between the white line and the orange line in Fig. 4-4 counts as a point. Are there any other points in Fig. 4-4? No, because there are no other intersections between two lines in the figure.

Given our definition of a point, are there any points in Fig. 4-2? No, because there are no intersections between the two lines in the figure.

Do the lines in Figs. 4-1, 4-3 and 4-4 have width? We cannot answer that question, because we don't know what the word *width* means in the logically possible imagined world we have just created, the world populated by regions, boundaries, lines, and intersection between lines.

Does the point in Fig.4-4 have length? As in the case of *width*, we cannot answer that question unless we define the concept denoted by the word *length*. Again, we are in a logically possible imagined world apprehended only through our mind's eye, not the world of our sensory experience apprehended through our physical eye, ear, nose, tongue, and skin. .

For exactly the same reason, we cannot answer questions about the *area* or *volume* of an object in the world that we have constructed so far, at least not until we have defined the concepts denoted by the words *length*, *width*, *area*, and *volume* with sufficient clarity and precision.

Our definitions of *line* and *point* use the terms *region*, *edge*, and *boundary*. Let us ask, do the *regions* in Figs. 4-1 to 4-4 have areas? We don't know, because though we have used the words *region*, *edge*, and *boundary* in our definitions, we have not defined the concepts that the words refer to. So for now, we must take region, edge, and boundary as ***undefined concepts*** (also called ***primitive concepts***) in the theory of lines and points we have just created.

4.2 Lines with Width and Points with Area

Do lines have width? Do points and lines have area? In §4.1, we took the position that we cannot answer these questions because we have not defined width and area.

The concepts of length and width are not defined in Euclid's *Elements* either. As a result, the concept of area also remains undefined. But, aswe saw earlier, Euclid axiomatically adopted 'no' as the answer to the above questions, and postulated that lines have no width; and points have no length or width, and hence no area.

But what if we take the opposite position? Let us try, by first defining the length of lines:

> LENGTH (DEF): The length of a line is the number of points it contains.

This means that if a finite line A has more points than line B, A is longer than B. But in Euclid, every finite line has infinitely many points. So the statement that A has more points than B does not make sense.

What if we set up the following axiom to solve this problem:

> **Axiom:** Every line of finite length has a finite number of points.

In Euclid, because points have zero length and zero width, adding points together does not create a line with non-zero length. Furthermore, no matter how close two points are, there is always at least one point between them. Hence, no two points are adjacent, that is, next to each other.

In the world of integers, numbers 3 and 4 are adjacent because there is no integer between them. So are integers 458 and 459: there is no integer between them. In contrast, 3 and 7 are not adjacent, nor are 458 and 464.

In the world of rational numbers, however 3 and 4 are not adjacent. For instance, 3.6 lies between 3 and 4. How about 3 and 3.1? Are they adjacent? No, because 3.13 lies between them. What about 3.13 and 3.14? They are not adjacent either because 3.136 lies between them.

So, points in Euclidean geometry are like rational numbers rather than like integers, because they cannot be adjacent. In the geometry that we are setting up now, we are treating points like integers, where two points can indeed be adjacent.

We now have two theories of geometry: the rational number geometry of Euclid's *Elements*, and the integer geometry that we are considering. Given this situation, it is legitimate to ask: Do the two theories constitute two distinct theories of geometry? Or are they not distinct? Do they yield distinct theorems, which are logically contradictory?

Let us take a look.

Take the concept of bisection in Euclidean geometry. Theorem 10 in Euclid says:

Every finite line is bisectable.

This means that every finite line can be cut into two lines which are congruent (= of the same length).

Is this a theorem in a rational number theory of geometry? How about in an integer theory of geometry?

> ### Exercise 1
> Task: Define BISECTION. Then figure out if the following conjecture is true:
>
> There exist lines which are not bisectable.
>
> If you think it is true, prove it.

If you are given two theories of the same thing, and asked if they constitute two distinct theories, or the same theory, engaging with Ex. 1 will help you think towards an answer to that question.

4.3 Parallel Lines

Intuitively, we judge the lines in Fig. 4-5 to be "parallel", and those in Fig. 4-6 to be not parallel.

Figure 4-5

Figure 4-6

Underlying this intuition are certain properties we attribute to parallel lines, all of which are satisfied by Fig. 4-5, but not Fig. 4-6:

Table 1

	Property	Fig. 4-5	Fig. 4-6
a.	Equi-distance: The distance between the two lines is invariant even when extended indefinitely on both sides,	**YES**	**NO**
b.	Non-intersectability: The lines never intersect even when extended indefinitely on both sides.	**YES**	**NO**
c.	Perpendicular intersection: An intersecting straight line perpendicular to both the lines is possible.	**YES**	**NO**
d.	Conservation of the sum of internal angles: If a straight line intersects the two lines, the sum of internal angles on the same side is two right angles.	**YES**	**NO**

In Euclidean geometry, the properties in Table 1 converge: any pair of straight lines that has one of these properties also has the others. Hence, we may take any of these properties as definitional of the concept of parallel, and derive the remaining properties as theorems.

In non-Euclidean geometries, it is possible for a pair of lines to have one or more of these properties but not the others.

The term "parallel" is conventionally used only in the context of straight lines. However, unless the definition specifies that the term is restricted to straight lines, we can take them as holding on curved lines as well. Consider, for instance, the following pairs of curved lines.

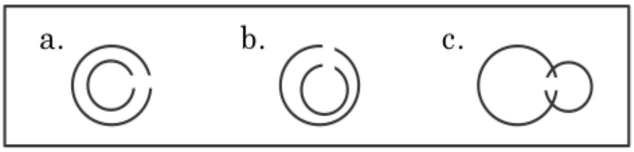

Figure 4-7

The pairs of lines in Fig. 4-7 are all circular (and therefore curved).

In Fig. 4-7a, the lines are concentric. Therefore:

- they are equidistant;
- they cannot intersect;
- a straight line drawn through the center would be perpendicular to both lines (provided we define *perpendicularity* and angle in a way that they apply to curved lines; and
- given any straight line intersecting the two lines, the sum of the internal angles on the same side is two right angle.

Thus, Fig. 4-7a exhibits all the properties in Table 1. Unless the definition restricts the concept of parallel to straight lines, the lines in Fig. 4-7a would count as parallel.

How about Figs. 4-7b and 4-7c? Using what is given in Table 2, work out how these two figures relate to the properties of parallelness.

Table 2

	Property	Fig. 4-7a	Fig. 4-7b	Fig. 4-7c
a.	Equi-distance	YES	NO	NO
b.	Non-intersectability	YES	YES	NO
c.	Perpendicular intersection	YES	NO	NO
d.	Conservation of sum of internal angles	YES	NO	NO

Whether or not the lines in Fig. 4-7b are parallel depends on how we define the term. If we take (b) in Table 2 as its defining property, then the lines in the figure are parallel, but not otherwise.

How about the lines in Fig. 4-7c? They are not parallel whichever property we choose as definitional of the concept parallel.

Exercise 2

Consider the following definitions of parallel lines:

Definition 1: Two lines A and B are parallel if and only if there exists a line C perpendicular to both.

Definition 2: Two lines A and B are parallel if and only if they are equidistant.

Draw figures to demonstrate that both these definitions are flawed. Then revise the definitions to remove the flaws.

In our discussion of parallel lines, the concepts of angle and of straight vs. curved lines turned out to be crucial. We are going to explore curved lines further in Unit 5. But in the meantime, it may be a good idea for you to invest some time to define straight line, angle, and vertex, and discuss your ideas with whoever is interested.

4.6 Summing up

In Unit 1–4, we used the methodological strategies of axiomatic inquiry in theory construction. An axiomatic system consists of premises (axioms and definitions), derivations, and conclusions. And the truth of a conclusion is proved by deducing it from the premises.

Mathematical theories are axiomatic systems. Theories outside of mathematics also have an axiomatic component. But in addition, they have another component, in which the predictions of the theory are matched against something outside the axiomatic component. Scientific theories, for example, also include an observational component. They are subject to the norm that the predictions (logical consequences derived from the premises) of the axiomatic component of the theory must fit with the observational generalisations (what is observationally established). Given that mathematical theories are about logically possible imagined worlds that we create, this norm does not hold on them.

Two important properties of the axiomatic mode of inquiry that we found in this Unit are:

A. In axiomatic inquiry, a question about something cannot be answered unless that something is defined clearly and precisely.

B. If we change an axiom or a definition, the truth of the conclusions may also change.

Finally, given the kind of logic used in mathematics, if we add a new axiom or definition to the system, what was established previously as true remains true. As we will see later, this is not the case with the axiomatic systems of scientific theories.

UNIT 5: A THEORY OF CIRCLES

5.1 Circles: as Regions vs. Boundaries

Let us go back to the concepts of regions and boundaries introduced earlier. Consider the two 'circles' in Figs. 5-1 and 5-2:

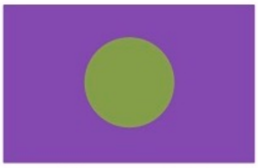

Figure 5-1

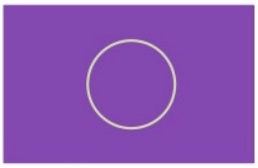

Figure 5-2

In Fig. 5-1, a purple rectangular region has a green circular region inside it. The green region has a boundary, also circular.

In Fig. 5-2, a purple rectangular region has a circular region inside it, but this time, this region is also purple, with a circular boundary.

We may now conceptualise 'circle' in two different ways:

Circle (CONCEPT 1): A circular region

Circle (CONCEPT 2): The boundary of a circular region

Euclid defines *Circle* as follows, which is the same as our CONCEPT 1:

Circle (CONCEPT 1: DEF): A circle is a plane figure bounded by one curved line, and such that all straight lines drawn from a certain point within it to the bounding line, are equal."

Euclid, *Elements*, Book I[1]:4

Given this definition, we may define *Circle* (CONCEPT 2) as follows:

Circle (CONCEPT 2: DEF): A circle is a curved line in which all straight lines drawn from a certain point within it to any point on the circle are equal.

This is a declarative definition that takes a circle to be an object. Instead, if we think of a circle (CONCEPT 2) as a trajectory of a point, we can define CONCEPT 2 as follows:

Circle (CONCEPT 2: DEF Alt): A circle is the trajectory of single point such that all straight lines drawn from a single point to the point in motion are equal.

In modern terminology, CONCEPT 1 is called a 'disk', and CONCEPT 2 is called a 'circle'. (See the Wikipedia entry on 'circle' at:
https://en.wikipedia.org/wiki/Disk_(mathematics))

Whether we are talking about CONCEPT 1 or CONCEPT 2, the point is the 'centre' of the circle, the boundary is called the 'circumference', and the straight line from the center to a point on the circumference is called the 'radius'. A straight line from one point on the circumference to a point on the opposite side, passing through the center, is called the 'diameter'.

Exercise 1

Prove that the length of the diameter of a circle is twice the length of its radius.

Something to think about:

Do circle as a boundary and as a region have length? Do they have areas?

As pointed out in Unit 4, we cannot answer these questions until we define the concepts of *region* and *boundary*. So, following Euclid's strategy, let us postulate our answers as axioms:

Axiom 1: Regions have area.

Axiom 2: Boundaries of regions are lines that have length, but no breadth or area.

We had asked if a point as an intersection between two lines has area. Axiom 2 above expresses our decision to attribute the property of length without breadth. If so, the intersection between two lines has neither breadth nor length.

That means that a circle as a circular line (CONCEPT 2) has length but no breadth; and a circle as a circular region (CONCEPT 1) has area.

Needless to say, we could have chosen to say that a circular line has both length and breadth. That is what we did in §4.2, where we chose to attribute the property of breadth to lines. Here we are taking a different track, and moving back to Euclidean geometry.

Exercise 2

TASK 1: Keeping in mind the Unit 1 discussion of the two ways of defining lines in terms of boundaries, propose a definition for 'circular line', such that:
(a) circle is a sub-category of circular line,
and then propose another definition for 'circular line', such that:
(b) circle is distinct from circular line.

TASK 2: Which of the two concepts in Task 1 would you regard as the better one? State your reasons.

Task 3: Within an axiomatic system which incorporates Axioms 1 and 2, try to come up with definitions of length, breadth, and area.

Instead of defining length or clarifying the concept of length through axioms, Euclid set up the concept of ***congruence*** to replace the notion of 'same length' (or equal length). In this system:

Two lines A and B are congruent iff one of them can be placed on top of the other such that they coincide exactly.

Suppose we do the following activity in our mind:

Draw a straight line A, and copy paste it as A'.

You would agree that A and A' would be congruent, and hence will have the same length. Likewise:

Draw a semi-circular line B, and copy paste it as B'.
Draw a wavy line C, and copy paste it as C'.

Lines B and B' will be congruent, and so will lines C and C'. The members of each pair would be have the same length. But what about the comparative lengths of A, B and C? How do we tell if they have the same length, or whether one of them has greater length than another?

We cannot answer that question, because we have not defined or clarified length in such a way that we can compare lengths of lines with distinct shapes.

Exercise 3

Within an axiomatic system which incorporates the definition of length, and Axioms 1 and 2 in §4.2, is it possible to check if two lines that are not congruent have the same length, or if one of them has greater length? Prove your answer.

5.2 Circle Theorems

You must have encountered the so-called circle theorems in school. [Remember theorems like: "A triangle in which one of the sides is the diameter of a circle, and every vertex is on the circle, is a right-angled triangle."] Circle theorems are about the correlation between circles and polygons. Many of them are correlations between circles and triangles. We will state a few of them as conjectures. Try to prove them to establish them as theorems. Resist the temptation to do an Internet search for a proof that someone else has done.

Let us begin with the idea of circumscription and inscription.

Conjecture 1:

For every triangle, there exists exactly one circle that circumscribes it.

Conjecture 2:

For every triangle, there exists exactly one circle that is inscribed in it.

To repeat what we hope is now part of your blood stream, we can neither prove nor disprove these conjectures without defining circumscription and inscription. Let us try these definitions:

Circumscription (DEF): A circle circumscribes a triangle iff every vertex of the triangle is on the circle.

Inscription (DEF): A triangle inscribes a circle iff the circle touches every side of the triangle.

These definitions have a conceptual flaw in that they don't let you see what is shared by the two concepts that the words *circumscribe* and *inscribe* refer to. Suppose we use the two words as inverses, and hold that:

A geometric figure A circumscribes a geometric figure B iff B is inscribed in A.

If we accept this view, we do not need two different concepts. We can use the word circumscribe in such a way that Y is inscribed in X means the same as X circumscribes Y. To illustrate, consider these diagrams:

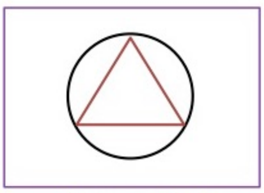

Figure 5-3

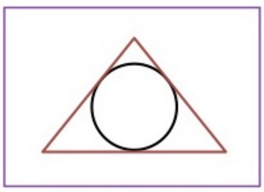

Figure 5-4

Given what we have said about what we want the words *circumscribe* and *inscribe* to mean, we may describe the relation between the circle and the triangle in these two figures as follows:

~ In Fig. 5-3, the circle circumscribes the triangle.
  (= The triangle is inscribed in the circle.)

~ In Fig. 5-4, the triangle circumscribes the circle.
  (= The circle is inscribed in the triangle.)

This conceptualisation of circumscription and inscription deviates from what you may have learnt in school. But here is the way the Wikipedia entry on inscription conceptualises it:

> "In geometry, an inscribed planar shape or solid is one that is enclosed by and "fits snugly" inside another geometric shape or solid. To say that "figure F is inscribed in figure G" means precisely the same thing as "figure G is circumscribed about figure F"." (https://en.wikipedia.org/wiki/Inscribed_figure)

If we adopt the Wiki conceptualisation, how do we define 'circumscribe'?

Exercise 4

TASK 1: Complete the following definition:

Figure X circumscribes figure Y iff …

Using that definition, prove or disprove Conjectures 1 and 2.

TASK 2: Take a look at the eight circle theorems at
http://www.timdevereux.co.uk/maths/geompages/8theorem.php

For your own practice, treat these theorems as conjectures and try to prove as many of them as possible. It is okay to draw upon your memory of the proofs that you are familiar with from school, but you must state them in terms of the structure of premises-derivation-conclusion (PDC) illustrated in the previous Units.

Resist the temptation to do an Internet search to find already available proofs, because that would rob you of a learning opportunity. To develop the capacity to come up with and write proofs with clarity and precision, what matters is the effort that YOU put into this task (either individually or in groups), not whether you are successful in coming up with a proof, or whether an expert judges it to be valid.

5.3 Circles and Regular Polygons

We are now ready to investigate an interesting question:

Can a circle be a regular polygon?

Notice that we are not asking whether a regular polygon can ***approximate*** a circle. We are asking if there are regular polygons that ***are*** circles. There is a world of difference between the two.

A brief introduction to the terminology first.

A polygon is ***equilateral*** iff its sides are of the same length.

A polygon is ***equiangular*** iff its angles are equal.

A polygon is ***regular*** iff it is equilateral and equianglular.

(Before going further, it may be useful to go to the Wikipedia entry on regular polygon at https://en.wikipedia.org/wiki/Regular_polygon and read the introductory paragraph, as well as the material under "General Properties".)

Now let us do a thought experiment.

Draw a regular polygon in your mind.
Inscribe a circle inside it.
Now circumscribe the polygon with a circle.

Gradually increase the number of sides of the polygon.

As the number of sides increases, you will see this in your mind's eye:

As the number of sides increases, the distance between the two circles decreases.

To prove that a circle is a regular polygon, you will have to prove that as the number of sides increases, the outer circle and the inner circle coincide, which means that they will also coincide with the polygon.

Can you prove or disprove the claim that it is possible to do so?

[Clue: To construct that proof, use the definitions of 'polygon' and 'circle'.]

Exercise 5

Think about the axioms and definitions we have so far.

In Euclidean Geometry, there is at least one point between any two points, however small the distance between them. This implies that every line, however short, has infinitely many points.

In the non-Euclidean Geometry of §4.2, points can be adjacent, and every finite line has a finitenumber of points.

To prove or refute the conjecture that every circle is a regular polygon;
Did you use a Euclidean system, or a non-Euclidian one?
Would your answer change if you use the other system?

5.4 Degrees of Freedom

For a clearer understanding of the two concepts denoted by the word *circle*, let us try a thought experiment.

Imagine a circular line suspended ten meters above the ground. This line does not form a complete circle; it is only four-fifths of a circle. For a circle to be complete, the line has to return to itself.

Imagine an ant walking along that line. The ant can go backwards and forwards on the line, but cannot step off the line. If she does — walks from one point on the line towards the center of the circle — then she would fall down. In this situation, we say that the ant has only one degree of freedom, namely, backwards and forwards. Even if the circular line is a complete circle, such that if the ant keeps walking it will return to where it started from, the degree of freedom would be the same.

Now imagine a circular disk suspended ten meters above the ground. The ant can now walk not only backwards and forwards along the boundary of the disk, but also backwards and forwards along any diameter line, without falling off the disk. In this situation, we say that the ant has two degrees of freedom.

Now imagine a spherical surface — the surface of a tennis ball. Imagine the ant walking on the surface of the ball. It cannot pierce through the ball to get to the center or to the opposite side. Studying the movement of the ant would be part of the geometry of spherical surfaces which are two-dimensional.

How about if the ball is not a tennis ball but a ball of rice? This time, the ant can burrow through the rice, and get to the center and to the opposite side. Studying the ant's movement here would be part of the geometry of a three-dimensional ball.

How many degrees of freedom does the ant have in these two cases? (Keep going till you feel the smoke coming out of your ears. ☺)

5.5 Summing up

A salient point that emerged in our discussion in Units 4 and 5 is the importance of conceptualising the meaning of an academic term in a theory before we try to define it. This means asking ourselves:

> What is the concept we need, and are looking for?
>
> Having decided on the concept, how do we define it?

This distinction came up quite starkly in our conceptualisation of lines and points in Unit 4 and of circles in Unit 5.

Such conceptual clarification — the process of conceptualising and defining — is important in all domains of inquiry and research. If you wish to assert the existence or non-existence of the soul, it is important to ask what you mean by 'soul' and figure out the meaning sufficiently clearly so that you can assert or deny its existence. Is the concept of 'soul' distinct from the concept of 'mind'? If yes, what is the difference? Does the soul exist after the death of the physical body? If it does, does it retain the mental properties of the person before death? If it does, does it wander the earth, or go to heaven, or hell? Does the soul have supernatural powers to influence the physical world? (Can a soul pick up a stone and hurl it at a window?)

Without reflecting on such questions and arriving at a decision, asserting or denying the existence of souls is meaningless. The next step would be to define the concept with as much clarity and precision as you can muster, such that you can discuss the issue of the existence of the soul with others, present arguments in support of or against its existence, and perhaps engage in a debate on its existence.

Another learning point in Unit 5 was the distinction between

the ***properties of an entity*** (e.g., of polygons, of circles)

and

the ***correlation between the properties*** of two (or more) entities (e.g., between circles and polygons).

A third point was the need to revise and refine the statements we encounter in textbooks or the classroom.

In all of this, you have been learning the art and craft of inventing and defining concepts, and coming up with and articulating conjectures and proofs.

UNIT 6: PULLING THE THREADS TOGETHER

6.1 Constructing and Evaluating Theories: the Art and the Craft

We expect that by now, you would have an intuitive understanding of how a theory is constructed. Let us state that understanding explicitly.

The first step when we construct a theory of X is to write down a set of statements that we already know about X. This is what we called a ***description*** of X.

Among the statements, we select some as premises, and others that we can derive as logical consequences of those premises, we take as conclusions. Having done this, we provide proofs for the conclusions, by demonstrating how they are derived. This is what we have called the **P**(remises)-**D**(erivation)-**C**onclusion structure.

We make this move with caution. It is tentative, because in trying to derive the conclusions from the premises, we may change their status. It may turn out that some of the statements that we initially viewed as conclusions cannot be derived from the premises, so we will need to take them also as premises. As a consequence, some of the statements that we had thought of as premises may become conclusions. You had a taste of this kind of a situation in §4.3, in our discussion of the definitions of parallel lines vs. the theorems that follow logically.

Having separated the statements into premises and conclusions, we further separate the premises, equally tentatively, into definitions and axioms. This sifting is necessary because it shapes our conceptualisation. However, it would not affect our proofs, since whether we take a statement as an axiom or a definition does not have any effect on our derivation.

The next step is to evaluate our proofs. Does each step in our derivation follow logically from the preceding steps and the premises? In other words, have our attempts at proofs given us valid proofs? If there are steps that make our proofs invalid, we need to modify our premises, or unearth statements that we have been implicitly assuming, and formulate them as explicit premises. This is likely to increase the number of premises.

Alongside adding our implicit assumptions as explicit premises, we might also add to our list of ***conjectures***. These are the statements that we think are true of X, but have not proved them. Every time we add a conjecture, we repeat the steps outlined above, so that the process is thorough.

As mentioned earlier, for theories outside of mathematics, we need one other component. Scientific theories, for example, require us to discover and establish observational generalisations, and show that

(i) our theoretical conclusions follow from (are correct predictions of) these generalisations, and

(ii) they can successfully explain the asymmetries that these generalisations point to.

Theories of other kinds, including conceptual theories, ethical theories, and theories of value require us to satisfy similar external conditions that involve equivalents of observational generalisations.

6.2 Abstracting

Geometry is a mathematical theory concerned with shapes, their relations and relative arrangements in space. It is ultimately rooted in our visual experience of the world. When we compare a sheet of A4 size paper on a table with a DVD next to it, we distinguish between the shapes of these objects as rectangular vs. circular. When we say rectangular and circular, we are focusing on the shape. The objects — the sheet of paper and the disk — have other properties such as colour, weight, rigidity, opacity, and so on. But geometry ignores these aspects, and extracts the relevant attributes of the objects into abstract entities. These ***abstract objects***, which we call *circle, rectangle*, and so on, do not have colour, weight, rigidity, opacity, and the like. So a spherical apple, for example, is edible, but a sphere is not.

Such abstraction, which isolates and studies properties of concrete objects in the world, and transforms them into abstract attributes or abstract objects, is not restricted to mathematical theories. Scientific theories are also built out of such abstractions.

6.3 Generalising and Specialising

Imagine this. We take a sample of ten polygons, and build descriptions of each of them. These descriptions would be the equivalents of what are called ‘data points’ in scientific theories. Based on these descriptions, we come up with a set of conjectures. We now examine three more polygons, and check if our conjectures hold on them. And then we examine five more polygons. We go on doing this until we can find no ***counterexamples*** to our conjectures. Having satisfied ourselves that there are no counterexamples, we can now proceed to prove the conjectures.

What we have described above is the ***process of generalisation*** from a sample of polygons to the population of polygons. This methodological strategy of generalising from a sample to a population is relevant in any form of collective inquiry, including scientific inquiry.

Needless to say, in the process of generalising from the sample to the population, we might discover counterexamples. When faced with counterexamples, we may either abandon our original conjecture as false, or see if it can be saved by restricting the scope of the conjecture to a sub-population. For instance, we might find that all our counterexamples involve polygons with concave angles, in which case we might restrict the scope of our conjecture to convex polygons. We may have to restrict the scope even further, to regular polygons, and so on. Such a move involves the ***process of specialisation***.

Both generalisation and specialisation are needed in theories outside mathematics as well. We might originally propose a conjecture on vertebrates, but we may find that it applies to animals in general, in which case we generalise it. Alternatively, we may discover counterexamples and may need to restrict the scope to mammals, in which case we specialise it.

6.4 Reasoning, Predicting, Explaining and Proving

As we have seen, theories of geometry are ***axiomatic systems***. This is true of all mathematical theories. In such a theory, a conjecture is ***proved*** by deriving it from the premises, using deductive ***reasoning***. So mathematical theories involve constructing knowledge through pure reasoning, combined with imagination, insight, and intuition.

Axiomatic systems are subject to the conditions of coherence. One of the conditions of coherence is the absence of logical contradictions internal to the system. In addition, coherence also includes the conditions of ***logical connectedness***, ***generality*** (widest range of conclusions), and ***simplicity*** (fewest possible premises — also called *parsimony*, or *Occam's Razor*).

Since scientific theories have an axiomatic component, they are also subject to the conditions of coherence. The scientific equivalents of mathematical theorems, as we have seen, are ***predictions***: the logical consequences deduced from the premises of the theory. But as pointed out earlier, scientific theories have the additional requirement that the predictions must agree with the observational generalisations. That translates as the requirement that not only must theories make predictions, but those predictions must be correct, where 'correct' means logically consistent with the observational generalisations.

An important condition that we must point out at this juncture is that it is not enough for the predictions to be corect. They must also explain the asymmetries in observational generalisations. In other words, scientific theories simultanously respond to the questions, "Is it true?" (rational justification), and "Why is it true?" (explanation, and the understanding that comes from that explanation). Explanation is central to scientific theories.

In the latter function, that of explanation, mathematical theories also don't just predict, but also explain the asymmetries in statements that are accepted as true.

Let us take a few examples. Here are a few statements that we take to be 'true statements' (TS). For each of them, we need to ask: "Why is this so?"

TS1: No triangle can have a reflex angle.
Non-triangle polygons can have reflex angles.

TS2: An equilateral triangle is an equiangular triangle and vice versa.
Non-triangle polygons do not exhibit this pattern,

TS3: An angle in a triangle cannot be varied without simultaneously varying the length of at least one of its sides.
Non-triangle polygons do not exhibit this behaviour,

Mathematical proofs that demonstrate the truth of the above statements also help us understand the asymmetries in TS1-3 by providing explanations for them.

6.5 Transdisciplinary Epistemology

The concepts and strategies of theory construction illustrated in this monograph were introduced in the specific context of geometry, though we occasionally demonstrated their usefulness in theory construction in biology.

These concepts and strategies are not restricted to geometry and biology: they play an important role in the construction and evaluation of theories in all domains of research, ranging from mathematics, and the physical-biological-human sciences, to the humanities. We will therefore refer to them as ***transdisciplinary*** concepts and strategies, using *trans-* in the sense that they cut across specialised disciplines, discipline groups and fields, and exist at a more abstract level.

To take an example of what we mean by transdisciplinary, the so-called *theory of classical mechanics* is specific to the highly specialised field that studies gravity and motion within the discipline called physics. However, the concept of ***theory*** is a trans-disciplinary one. It is in this sense that we talk about transdisciplinary concepts.

Take another example. The concepts of gravity, velocity, and acceleration are specific to physics, and the concept of skeletal structure, and of cells being composed of biomolecules, is specific to biology. But the concepts of ***compositionality***, ***structure***, and very concept of ***concept*** are transdisciplinary.

In §1.5.3, we made a distinction between ***axioms*** that are specific to a particular field or discipline, and ***general axioms*** that are part of all academic knowledge. General axioms are transdisciplinary. We may make the same distinction among definitions. The definition of force as *that which causes a change in the velocity of an inanimate entity* is specific to Physics. But the definition of force as *that which causes change* is a transdisciplinary definition, extendable to the animate, even human domains.

To repeat what we said in §3.3:

Transdisciplinary relations that appear in theories across disciplines include:

- Subcategorization: *x* is A SUBCATEGORY OF *y*.
- Compositionality: *x* is COMPOSED OF *y, z, …* (Variants: is MADE UP OF / is A CONSTITUENT OF, is DECOMPOSABLE INTO, …)
- Ordering: *x* is ORDERED PRIOR TO *y*. (Variants: is RANKED HIGHER THAN; PRECEDES)
- Logical consequence: *x* is a LOGICAL CONSEQUENCE OF *y*.
- Logical contradiction: *x* LOGICALLY CONTRADICTS *y*.
- Equality: *x* IS EQUAL TO *y*. (Variant: IS EQUIVALENT TO, IS AN ANALOGUE OF, IS A HOMOLOGUE OF, …)
- Correlation: *x* CORRELATES WITH *y*.
- Causation: *x* CAUSES *y*
- Instantiation: *x* is AN INSTANCE OF *y*. (Variants: is A MEMBER OF set/category y, is AN EXAMPLE OF y, is A SAMPLE OF y, …)
- Negation: x is THE NEGATION OF *y*. (Variant: is THE OPPOSITE OF.) (Also see logical contradiction)

We hope that the journey through geometry that we have undertaken so far has given you a glimpse into the transdisciplinary epistemology of academic knowledge, and has planted seeds that can develop the capacity to engage in research in any domain of academic knowledge.

6.6 Admissible Sources of Knowledge

The first paragraph of the introductory chapter of the book, *The Works of Archimedes*, by T L Heath begins as follows:

> "A LIFE of Archimedes was written by one Heracleides, but this biography has not survived, and such particulars as are known have to be collected from many various

> sources. According to Tzetzes he died at the age of 75, and, as he perished in the sack of Syracuse (B.C. 212), it follows that he was probably born about 287 B.C."

Notice the use of "it follows that," in the second sentence. The phrase signals the strategy of ***reasoning*** that forms of the core of ***rational inquiry***. We have discussed the methodological strategies of rational inquiry at length in the previous Units. Let us state the premises and the conclusion of the passage explicitly:

> Premise 1: Archimedes died at the age of 75
>
> Premise 1: Archimedes died in 212 BCE.
>
> Conclusion: Archimedes was born in 287 BCE.

Is this conclusion true?

We can answer that question as follows:

> If premises 1 and 2 are true, and the derivation of the conclusion from the premises is valid, then it is true that Archimedes was born in 287 BCE.

This answer tells us that there are two conditions for our accepting the truth of the conclusion:

> Condition A: The premises must be true.
>
> Condition B: The derivation of the conclusion from the premises must be valid.

How do we know that the premises are true? The answer is:

> Premises 1 and 2 are asserted by Tzetzes.

But why should we believe that what Tzetzes said is true? An Internet search for the name Tzetzes leads us to John Tzetzes, a Byzantine poet and grammarian who is known to have lived at Constantinople in the 12th century. T L Heath is making the assumption that John Tzetzes' ***testimony*** is a reliable ***source of knowledge*** to conclude that Archimedes was born in 287 BCE.

Relying on written testimonies by previous authors is an important methodological strategy of human history, in domains where written records are available. A similar methodological strategy is used in trials in the criminal court: spoken testimonies of eye witnesses and of expert witnesses are taken as reliable sources of knowledge.

You can see that spoken and written testimonies are not admissible as sources of knowledge in the physical and biological sciences. So, the fact that *Isaac Newton says*:

> The gravitational attraction between two bodies is directly proportional to the product of their masses, and indirectly proportional to the square of the distance between them,

does not allow us to conclude that the proposition:

> "The gravitational attraction between two bodies is directly proportional to the product of their masses, and indirectly proportional to the square of the distance between them,"

is true.

Similarly, the fact that *Charles Darwin says*:

> All existing and extinct life forms on the earth evolved from unicellular ancestors,

does not allow us to conclude that the proposition:

> "All existing and extinct life forms on the earth evolved from unicellular ancestors,"

is true. Expert testimonies are admissible sources of knowledge in criminal trials and in human history, but not in the physical and biological sciences.

What constitutes an admissible source of knowledge in scientific inquiry? The answer is: observational reports. An observational report is an eyewitness testimony.

6.7 The Nature of Truth in Mathematics

What is the nature of premises in mathematics? What are the kinds of premises that are accepted as sources of knowledge in mathematics?

In §4.2, we saw an example of the contrast between the Euclidean axiom that every finite line, however small, has infinitely many points, and the non-Euclidean axiom that every finite line has a finite number of points, such that the length of a line is the number of points it contains. If we adopt the Euclidean axiom, we deduce the conclusion that every line is bisectable. But if we adopt the alternative axiom, we deduce the conclusion that there exist lines which cannot be bisected.

These theorems are logically contradictory. So, if they were from the same theory, rationality demands that we reject at least one of them as false. But unlike the axioms (laws, constraints) in science, truth and falsehood are not properties of mathematical axioms. Why is that so?

This is because scientific theories are about the particular world we live in, but mathematical theories are about logically possible imagined worlds. And there can be many such worlds. So all that we can say is this:

> In a world in which the Euclidean axiom of the number of points in a line is true, the theorem that every line is bisectable is true.

But: In a world in which the non-Euclidean axiom of the number of points in a line is true, the theorem that there exist lines that cannot be bisected is true.

There is no logical contradiction between the two because they are about two different worlds.

Similar remarks apply to the geometry of flat surfaces and of spherical surfaces. You might have heard that Euclidean two-dimensional geometry is a geometry of flat surfaces, while Riemannian two-dimensional geometry is a geometry of spherical surfaces. The difference between them is:

IN A FLAT SURFACE GEOMETRY	IN A SPHERICAL SURFACE GEOMETRY
No straight line, regardless of how far it is extended, can meet itself.	Every straight line when extended meets itself.
No two straight lines can intersect at two distinct points.	Any two straight lines when extended intersect at two distinct points.
The sum of angles in a triangle is two right angles.	The sum of angles in a triangle is more than two right angles, and can be upto three right angles.

All this shows that the truth of a mathematical theorem is relative to the theory that it is a part of. Hence, mathematical truths are of the form:

> If such and such premises are true, such and such conclusions are also true.

Are the premises true? Mathematics has nothing to say about that. This is a fundamental difference between mathematical and scientific truths.

UNIT 7: CONSTRUCTING THEORIES IN BIOLOGY

7.1 Beyond Geometry: Theory in Biology

In Units 1–6, we outlined some of the methodological strategies for constructing and evaluating theories, where we used geometry as our primary terrain to exemplify and practice theory building. In this unit, we will move to a different terrain for exemplification and practice — the terrain of biology, but using precisely the same methodological strategies. Needless to say, the migration from mathematics to science requires an additional norm: *the logical consequences of a scientific theory must not only correctly predict the observational generalisations relevant to the theory, but also explain the asymmetries in those generalisations.* (§6.4)

We hope that this migration from geometry to biology would shed light on those transdisciplinary modes of thinking and reasoning that are ***transferrable*** from one domain of knowledge to another. With that purpose in mind, this unit will focus on two different theories in biology:

a theory of *anatomy; and*
a theory of *habitat.*

These theories, combined with theories of biological *function*, biological *change,* and biological *development* are essential to the construction of a theory of biological *evolution*. But the challenges of constructing each of these theories is beyond the scope of this monograph.

7.2 Observing, Reasoning, and Two Formalisms for Reasoning

Suppose you look out of a window and see what is given in this photograph.

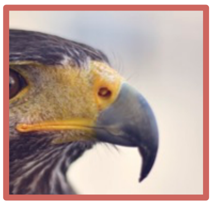

Figure 7-1

Think of the border of the photograph as the frame of the window. You see only a part of the entity, not the whole entity. Will you be able to answer the following multiple choice questions?

1. Is the entity whose part you are looking at:
   a. inanimate, animate, or neither?
   b. a man, a woman, a child, or none of these?
   c. an animal, a plant, or neither?
   d. a frog, an insect, a tree, a snake, a bird, a bat, or none of these?
   e. a hawk, an eagle, a parrot, a crow, a humming bird, a book, or none of these?

Write down your answers to these questions. And now answer the following 'YES-NO' questions. It would be a good idea to write down your answers.

2. Does this entity have the following? Put a Y(es) or N(o) in the box to the right of each.

a.	legs		e.	blood		i.	a tail		m.	fur	
b.	lungs		f.	bones		j.	a beak		n.	fins	
c.	hair		g.	cells		k.	nostrils		o.	fingers	
d.	heart		h.	claws		l.	eyes		p.	feathers	

3. Write down your answers to these questions as well:
   a. If your answer to question (1a) is yes, how many?
   b. If your answer to (1q) is yes, how many?
   c. What else can you say about this entity?

If you now compare your answers with the answers from a random collection of high schoolers, you will find something surprising: almost all of themtheir answers would be the same as yours. And interestingly, all this just because you looked at the picture in *Fig. 7-1*, and recognised it as a beak (question (2j)).

Now, only two, or perhaps three of your answers (the answer to (2j, k, l) came from your ***observation***. The rest came from your ***reasoning***.

To arrive at an answer through reasoning, you used a set of statements of the form:	If an entity has X, then it ……
Logicians refer to statements of this form as ***conditionals***:	If X, then Y.
To express conditionals with a symbol in formal logic, logicians use an arrow: It expresses the relation of ***implication*** (X implies Y).	X → Y
In the physical sciences, the corresponding relation is that of ***equality***, expressed by the symbol '='	X = Y
The equality symbol is equivalent to a double headed arrow (if and only if) in logic.	X ←→ Y

In the formulation of laws in the sciences of inanimate entities (i.e., the physical sciences), the most common relation is that of equality. For the formulation of the laws in the sciences of animate entities (i.e., biological sciences), we will rely instead on the relation of implication.

7.3 A Theory of Parrots

Notice that the questions in (1) are about the ***categorization*** of an entity, one part of which you can observe. In contrast, the questions in (2) and (3) are about the ***correlations*** between biological traits. Based on these correlations, you can infer information about one part of the organism from information about another:

4. a. If the entity has a beak, then ……
   b. If an entity is a parrot, then ……

Chances are that your answer to question (1e) is that the picture in *Fig. 7-1* is that of a parrot. Needless to say, having a beak is not sufficient for you to infer that the entity is a parrot. To infer this, you also need some other features that you observed in the photograph.

The set of ***if-then conditionals*** on the basis of which you gave answers to (2) and (3) constitutes a ***theory*** of the anatomical properties of the taxon we call *parrot*. But notice that it also is the same as a theory of the anatomical properties of the taxon we call *bird*.

7.4 Theories of Birds and Bees

In Units 1-6, we illustrated one of the methodological strategies of theory construction: we construct a theory of X by taking a ***description*** of X and converting it into a Premise-Derivation-Conclusion structure; treating some of the description propositions as ***premises,*** others as ***conclusions***; and deriving the conclusions from the premises through ***deductive reasoning***. In §7.3, we extended that strategy to construct a rudimentary theory of the anatomy of parrots.

This theory begins with the premises in (4), now filled in (in (5)), and proceeds to a number of other premises of the form: If X, then Y:

5. a. If the entity has a beak, it is a bird.
  b. If an entity is a parrot, it has feathers.
  c. If an entity is a parrot, it has two legs.
  d. If an entity is a parrot, it has two wings.
  and so on.)

[Note that the statement, "If the entity has a beak, it is a parrot." is false.]

This is not a particularly interesting theory because it postulates a separate premise for each property of the bird. Remember we saw some conditions on axiomatic systems in §6.4? One of the conditions was the logical connectedness (of premises), and other conditions were that we minimise the number of premises (simplicity), and maximise the number of conclusions (generality).

And remember the idea of the logical inheritance of properties?

  The properties of a category are inherited by their subcategories. (§1.4; §3.2))

Given the general principle that *subcategories inherit the properties of their mother categories*, one way of minimising the number of premises is to state the conditionals on higher level categories. Thus, instead of (5c), we may postulate (6):

6. a. If an entity is a parrot, then it is a bird. (= 'Parrot' is a subcategory of 'bird'.)
  b. If an entity is a bird, then it has two legs. (= Birds have two legs.)

We can now deduce all the properties of parrots we have described above from parrots being a subcategory of birds. This allows for a powerful reduction of premises, because from a single set of axioms on birds, we can now predict the anatomical properties of not only parrots, but also hawks, eagles, vultures, crows, ravens, sunbirds, and so on,when combined with the subcategory statements in (7):

7. a. Parrots are a subcategory of birds.
  b. Hawks are a subcategory of birds.
  c. Eagles are a subcategory of birds.
  d. Vultures are a subcategory of birds.
  e. Crows are a subcategory of birds.
  f. Ravens are a subcategories of birds.
  g. Sunbirds are a subcategory of birds.

Given these subcategory statements, there is no need to duplicate the statement of properties for each of the subcategories separately.

Now, if you compare the properties of birds with the properties of snakes, reptiles, fish, frogs, and mammals, you will find that they have a number of properties in common. Such shared properties across taxa are called ***homologies*** in biology. For instance:

8.	a.	Snakes have vertebrae.	Fish have vertebrae.	Mammals have vertebrae.
	b.	Snakes have blood.	Fish have blood.	Mammals have blood.
		Snakes have mouth.	Fish have mouth.	Mammals have mouth.
		Snakes have eyes.	Fish have eyes.	Mammals have eyes.
		And so on.	And so on.	And so on.

We can achieve even greater economy of premises with the following statements:

9. Vertebrate (DEF): The category of organisms with vertebrae.

10. a. Birds are a sub-category of vertebrates.
b. Snakes are a sub-category of vertebrates.
c. Fish are a sub-category of vertebrates.
d. Reptiles are a sub-category of vertebrates.
e. Frogs are a subcategory of vertebrates.
f. Mammals are a subcategory of vertebrates. And so on.

Needless to say, the different subcategories have not only ***similarities,*** but also ***differences***. That is what the concept of homology implies: similarities among differences, unity in diversity, invariance under variability, and the like. We account for both similarities and differences by postulating the similarities on the mother category, and the differences on the daughter category. That leads to the tree structure of biological categorisation:

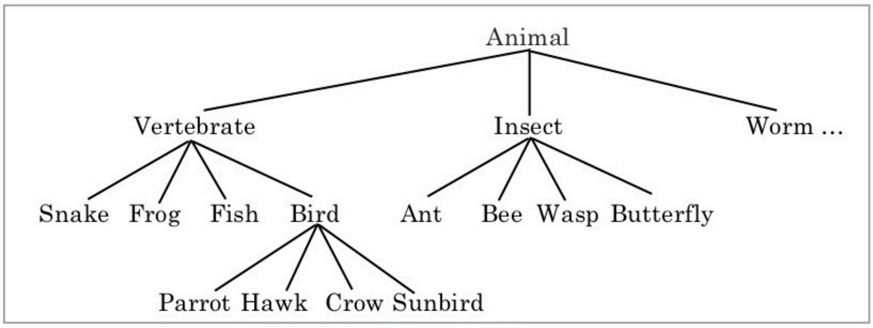

Figure 7-2

To take an example, the wikipedia entry on butterflies gives us a list of the properties of butterflies. (https://en.wikipedia.org/wiki/Butterfly) .

Adult butterflies
- have large, often brightly coloured wings
- have conspicuous, fluttering flight
- have a four-stage life cycle
- undergo complete metamorphosis
- lay eggs on the food plant, on which they grow into caterpillars, which
    - feed on the plant and
    - metamorphose into butterflies

This is just a small list. The entry does not mention that butterflies

- have compound eyes
- have six legs,
- have a neural system
- have a food canal
- have eukaryotic cells, and
- are multicellular

The reason why such properties are not mentioned in the entry is because these are predictable from the classificatory tree of categories and subcategories — a theory of the category of butterflies as a sub-theory of the category of animals, which in turn is a sub-theory of the category of animate entities.

Exercise 1:

TASK 1: Read the Wikipedia entries on snakes, frogs, and fish to make a list of the anatomical properties of each. Derive as many of them as possible by setting up the shared properties as properties of vertebrates. Those that are not shared, state as properties of the daughter categories.

TASK 2: Read the Wikipedia entries on bees, butterflies, insects, worms and animals to make a list of their anatomical properties. State properties of animals to deduce the shared properties of the daughters from the properties of the mother.

If you now go back to Unit 6, you will see that the methodological strategies we have used in §7.2 - §7.4 are those of description, abstraction, categorisation, generalisation, reasoning, and prediction. It is some of these strategies that you pratised in *Ex. 1*.

In the tree diagram in *Fig. 7-2*, we have used yet another strategy, that of representations. Having seen such visual representations in Venn diagrams (rectangles and circles), and in geometry (diagrams of straight and curved lines, triangles, rectangles, circles, …), it should be easy to connect the different kinds of representations in academic knowledge, and reflect on the IDEAS that the representations express (or are intended to express).

Before proceeding, it may be a good idea to reflect on this question: Do circles and rectangles in Venn Diagrams represent the same concepts as circles and rectangles in geometry? Your answer is likely to be no. If so, it would be a good idea to reflect on what exactly the conceptual differences between them are, and what the conceptual similarities are, and write down your answers.

Exercise 2:

TASK: Read the Wikipedia entries on unicellular life forms, prokaryotes, and eukaryotes, and construct a theory of the anatomical properties of life forms. To do this, you will need to incorporate the categories of animate entities, unicellulars, eykaryotes, and prokaryotes into the tree in *Fig. 7-2*.

7.5 A Theory of the Anatomy of Animate Entities

The theory that we have constructed so far is of the ***anatomical properties of animate entities***. It says very little about anatomical structure as such.

For instance, for a ***theory of anatomical structure,*** it is not sufficient to say that vertebrates have legs. We also need to say something about the structure of legs. What does this mean? Well, a human leg is composed of the upper leg, the lower leg and the foot. A human arm is composed of the upper arm, the lower arm, and the hand. We can unify legs and arms by stating these structural properties as the properties of limbs. We also need to say something about hands being composed of the palm and fingers, and something similar about feet. Finally, we need to say something about the three part structure of the digits on the hand, and also, the number of digits.

Constructing a theory of the anatomical structure of animate entities as a project is likely to take a few months. And if we want to make it comprehensive, perhaps two or three years. If one is adventurous, this would be a great challenge for a Master's or PhD thesis.

7.6 A Theory of Fitness: Survival and Extinction

Central to Darwin's theory of biological evolution is the idea of ***Natural Selection***, which he treats as being synonymous with the idea of ***Survival of the Fittest***. The term 'Survival of the Fittest' was coined by Herbert Spencer, famous for his idea of social Darwinism that says that superior physical force has a hand in shaping human history. Darwin adopted the phrase as an alternative to 'natural selection' in the fifth edition of the *Origin of Species*, published in 1869, intending it to mean "better designed for an immediate, local environment." He says:

> " ...can we doubt (remembering that many more individuals are born than can possibly survive) that individuals having any advantage, however slight, over others, would have the best chance of surviving and of procreating their kind? On the other hand, we may feel sure that any variation in the least degree injurious would be rigidly destroyed. This preservation of favourable variations, and the destruction of injurious variations, I call Natural Selection, or the Survival of the Fittest. Variations neither useful nor injurious would not be affected by natural selection, and would be left either a fluctuating element, as perhaps we see in certain polymorphic species, or would ultimately become fixed, owing to the nature of the organism and the nature of the conditions." (p.93)

To understand what Darwin says in this excerpt, let us formulate two related laws/axioms of Natural Selection, which we will call ***Survival of the Fittest*** and the ***Extinction of the Unfit*** respectively. In these statements, 'species' includes 'varieties.'

11. ***Fitness***: There are various degrees of fitness.

   A) ***Survival of the Fittest***: Only those species that are *fitter than all other species* are selected for survival. All other species become extinct.

   B) ***Extinction of the Unfit***: There is a *threshold* below which a species is unfit. Species that are unfit become extinct. All other species survive, regardless of their *relative fitness.*

Whether 'fitness' is elaborated as in (A) or (B), it is clear that we need a theory of fitness to critically evaluate and compare the predictions of (A) and (B).

Given these concepts of fitness, survival and extinction, what a theory of fitness needs to explain and predict are the ***asymmetries*** of survival and extinction. To illustrate, take yeast. The first yeast emerged on earth hundreds of millions of years ago. Various taxa of yeast still survive on earth, in contrast to many taxa which have become extinct. A theory of fitness ought to explain this asymmetry between survival and extinction.

If we expand the scope of the theory of survival and extinction from macro biology to micro biology, the theory needs to explain and predict the asymmetry between the survival, for instance, of the gene called cdc2 and the extinction of those genes that no longer exist today. This asymmetry between survival and extinction is relevant not only to molecules and taxa, but also cells, tissues, and organs.

In what follows, our focus will be on a theory of Extinction, as part of the theory of Natural Selection.

7.7 A Theory of Viability Selection and Habitat

The research literature on fitness talks about two kinds of selection, namely, ***fecundity selection*** and ***viability selection***. (e.g., "Role transformation of fecundity and viability: The leading cause of fitness costs associated with beta-cypermethrin resistance in Musca domestica" at (https://journals.plos.org/plosone/article?id=10.1371/journal.pone.0228268))

Fecundity is a matter of the number of offspring that an organism, variety, or species produces in the reproductive process. Viability, on the other hand, is a matter of continued existence from the conception till the organism/variety/species reach the reproductive stage. If a unicellular zygote dies before it develops into an embryo, if the embryo dies before birth, or if the infant dies before it becomes capable of reproduction, the issue of fecundity selection becomes irrelevant.

Given this situation, we will choose viability selection as the crucial theoretical concept on which we will construct our theory of fitness. We will also choose extinction (Idea B) as our criterion for the testing of predictions. Given that during the last hundred years, a large number of species have become extinct, it should be easy to test the predictions of the theory in terms of the data available on species that have become extinct during the recent years, and continue to become extinct. We do not need to count the number of offspring an organism or a variety produces.

With those remarks, let us define fitness in terms of the concepts of adaptation and habitat as follows:

12. ***Fitness*** (DEF): For a species/variety to be fit, its anatomy, physiology, and behaviour must be adapted to (aligned with) its habitat; a species/variety is unfit iff it is ill-adapted to (misaligned with) its habitat.

13. ***Law of Extinction***: A species/variety that is unfit becomes extinct.

The next step is to specify the parameters relevant to the fit between the anatomy, physiology and behaviour of a variety/species on the one hand, and the habitat.

For this, let us define habitat as follows:

14. ***Habitat*** (DEF): The habitat of an organism/variety/species X is the local three dimensional space in which X exists.

Let us take a few examples.

15. Imagine the habitats below, each of them being a space enclosed in glass:

	filled with	SURVIVAL / EXTINCTION		
		Fish	Mice	Earthworm
A	water, but no air or soil	survive	die	die
B	air. but no water and soil	die	survive	die
C	moist soil but no water or air	die	die	survive
D	moist soil below and air above the surface	die	die	survive

In order to provide theoretical explanations for such *experimentally testable correlations* in organisms(/varieties/species), *between*

a) an organism and the space surrounding it, on the one hand, and
b) its extinction on the other,

we need to understand

i) its anatomy, physiology and behavior, and
ii) the properties of the space surrounding it, and what exists in that space.

For instance, what are the anatomical and physiological properties that make fish survive in habitat A but not in B-D? What are the cellular and molecular bases of these macro level anatomical and physiological traits? These are some of the types of questions that would need to be answered for a coherent theory of fitness that makes testable predictions in why certain varieties/species become extinct in some habitats, and why some become extinct globally.

The parameters of water, air, surface of the earth, and under the earth as broad categories of habitat are not sufficient. For each of these, we also need to specify:

16. a. Temperature: The upper and lower boundaries for the range of temperature in which the species/variety can survive. (Bacteria can survive in temperature ranges in which humans and mice would not survive.)

b. Composition: The molecular composition of the substances. (Oxygen, Nitrogen, Carbon Dioxide in the air, water and soil)

c. Nutrients: Availability of nutrients (Plants for cows; animals for lions…)

Exercise 3:

TASK: Read the Wikipedia entries on the habitats of as many taxa as you can deal with, and construct a theory of fitness in terms of entity-habitat pairing that specifies the threshold below which the entity becomes extinct.

This task may take three hours, three months, or three years, depending on whether it is a learning task for an undergraduate course, a higher level research project , or a doctoral thesis.

If you are adventurous enough to pursue the task in *Ex. 3* as a topic for research, here is a hint. In order to construct a rigorous theory in a rigorous fashion, you would need appropriate logical and mathematical formalisms. We have already mentioned the formalism of formal logics for scientific laws formulated as if-then conditions in systems of formal logic, but we also need the representational system of Graph Theory. The one used in *Fig. 7-2* is that of a tree diagram, which is a visual representation in a Directed Acyclic Graph (DAG) in Graph Theory in mathematics. We are going to need these two in the construction of the theory of fitness as well.

In addition, a PhD student would do well to explore representational systems that computer scientists call Attribute-Value model.

(see: https://en.wikipedia.org/wiki/Entity-attribute-value_model)

To think about: At the end of §7.4, we said that §7.2– §7.4 have crucially used the methodological strategies of description, abstraction, generalisation, reasoning, prediction, and representation. Did we miss something?

Having gone through §7.5 and §7.6 as well, what do you think is the methodological strategy that we have used in these sections?

7.8 Levels and Dimensions of the Tree of Life

The Tree of Life in *Fig. 7-2* is that of taxa of animate entities based on anatomical properties. The discussion of the habitats of taxa lead to the possibility of a different system of categorization based on the similarities and differences in their habitats.

Take what the Wikipedia entry on water birds says:

> "A water bird, alternatively waterbird or aquatic bird, is a bird that lives on or around water. In some definitions, the term water bird is especially applied to birds in freshwater ecosystems, although others make no distinction from seabirds that inhabit marine environments. Some water birds (e.g. wading birds) are more terrestrial while others (e.g. waterfowls) are more aquatic, and their adaptations will vary depending on their environment. These adaptations include webbed feet, beaks, and legs adapted to feed in the water, and the ability to dive from the surface or the air to catch prey in water. "

Let us generalise the term 'water bird' to include to 'water life forms' to include all animate entities. If we make this move, the discussion in the previous section points to the following categorization:

17. In the *three dimensional space surrounding them* in their habitat, nimate entities require (a) the presence, and (b) the absence, of:
   i. water   ii. air   iii. soil

Suppose we use the symbol '+' for 'require the presence of' and '–' for 'require the absence of'. Using this notation, we may categorise animate entities as:

18. a. Water: [+W] or [–W]
   b. Air: [+A] or [–A]
   c. Soil: [+S] or [–S]

What we have in (18) is an attribute-value system. (see https://en.wikipedia.org/wiki/Entity-attribute-value_model). In this system, 'water', 'air', and 'soil' are 'attributes': each of them a dimension, an axis, or a parameter of categorisation, with '+' and '–' as the values of an entity along a given parameter.

Given this system, we may specify the values of the aspects of their habitat for fish, bird, mouse, earthworm, and frog as follows:

19. a. Fish $\begin{pmatrix}+\mathrm{W}\\-\mathrm{A}\\-\mathrm{S}\end{pmatrix}$ b. Bird $\begin{pmatrix}-\mathrm{W}\\+\mathrm{A}\\-\mathrm{S}\end{pmatrix}$ c. Mouse $\begin{pmatrix}-\mathrm{W}\\+\mathrm{A}\\-\mathrm{S}\end{pmatrix}$ d. Frog $\begin{pmatrix}+\mathrm{W}\\-\mathrm{A}\\-\mathrm{S}\end{pmatrix}$ **OR** $\begin{pmatrix}-\mathrm{W}\\+\mathrm{A}\\-\mathrm{S}\end{pmatrix}$ e. Earthworm $\begin{pmatrix}-\mathrm{W}\\+\mathrm{A}\\-\mathrm{S}\end{pmatrix}$ OR $\begin{pmatrix}-\mathrm{W}\\-\mathrm{A}\\+\mathrm{S}\end{pmatrix}$

A word about the terms 'water', 'air', and 'soil', and what the notation means in this attribute-value system:

20.	Must be surrounded by	Must not be surrounded by
a.	[+W]: "water"	[–W]: "water"
b.	[+A]: "air"	[–A]: "air"
c.	[+S]: "soil"	[–S]: "soil"

Now, the term 'water' may refer either to:

(i) water molecules (H_20); or

(ii) the substance we call water, with three physical states: gas, liquid, solid; or

(iii) water in the liquid state.

In our notation here, it means 'water as a substance in the liquid state'. This is because, for instance:

21. Fish cannot survive if there is air around them, even if the air has water molecules in it. Nor can they survive if the water is in the solid or gas state.

Aeroponic plants cannot survive without water as an aggregate of H_2O molecules, but they can survive without water as a substance in the liquid form (https://en.wikipedia.org/wiki/Aeroponics)

Do we need to specify every aspect of the habitat for every species/variety? Take water molecules, and the following universal law on animate entities:

Law 1: All life forms, in order to survive, need water molecules as part of their composition.

Let us use the symbol [WM] to denote the dimension of the water molecule. We may now specify the property that comes from Law 1 as:

22. a. Fish b. Bird c. Mouse d. Frog e. Earthworm

$$\text{a. } \begin{pmatrix} +\mathrm{WM} \\ +\mathrm{W} \\ -\mathrm{A} \\ -\mathrm{S} \end{pmatrix} \quad \text{b. } \begin{pmatrix} +\mathrm{WM} \\ -\mathrm{W} \\ +\mathrm{A} \\ -\mathrm{S} \end{pmatrix} \quad \text{c. } \begin{pmatrix} +\mathrm{WM} \\ -\mathrm{W} \\ +\mathrm{A} \\ -\mathrm{S} \end{pmatrix} \quad \text{d. } \begin{pmatrix} +\mathrm{WM} \\ +\mathrm{W} \\ -\mathrm{A} \\ -\mathrm{S} \end{pmatrix} \textbf{OR} \begin{pmatrix} +\mathrm{WM} \\ -\mathrm{W} \\ +\mathrm{A} \\ -\mathrm{S} \end{pmatrix} \quad \text{e. } \begin{pmatrix} +\mathrm{WM} \\ -\mathrm{W} \\ +\mathrm{A} \\ -\mathrm{S} \end{pmatrix} \textbf{OR} \begin{pmatrix} +\mathrm{WM} \\ -\mathrm{W} \\ -\mathrm{A} \\ +\mathrm{S} \end{pmatrix}$$

However, all the organisms classified in (22) are subcategories of animate entities. So, if we assign the property to animates, the mother category, as follows:

23. Animates: [+WM]

there is no need to specify it on each of its subcategories. Given this specification on the category ‘animate’, its daughters and other subcategories will inherit this assignment of attribute value.

What all this points to is a need to construct a classificatory system for animate entities involving attributes and values of the kind illustrated above. That system would complement the system illustrated in *Fig. 7-2*.

Exercise 4:

TASK: Read the Wikipedia entries on the habitats of as many taxa as you can deal with, and construct a classification of animals along the lines indicated above, including the specification of temperature, nutrients, and so on.

As a learning task or a research project, this one is much like the one in *Ex. 3*.

7.9 Summing up

In a recent article in the journal *Nature*, Nobel Laureate Paul Nurse points out that biology needs to go beyond mere data and descriptions to ***ideas*** that can explain and predict what we find in the data and descriptions.

Paul Nurse, "Biology must Generate Ideas as well as Data."
Downloadable at: https://www.nature.com/articles/d41586-021-02480-z

What we have done in this unit may be viewed as a way of following Nurse’s recommendation, by demonstrating what it takes to generate ideas and develop them as theories whose predictions are testable.

With that remark, we will deliberately leave the rest of the "Summing-Up" to you. What we would like you as a reader to do is to go through all the units, and write a summary of the monograph. When you do that, make sure to specify what you learnt from it that is of value to you, and will be of value to you in your future work.

Finally, we have a request. Please send us that summary at tara.mohanan@gmail.com. We would like to find out whether or not our efforts have benefited you, and how we can revise the monograph to benefit more learners, and provide higher value.

Unit 8: Geometry as Science and as Math

8.1 Introductory Remarks

This unit, we hope, will provide an experiential understanding of the distinction between mathematical and scientific theories. To this end, we explore two-dimensional Euclidean Geometry, first as an observational science, and then as an axiomatic system in mathematics. That exploration leads to a somewhat unconventional conceptualisation of two-dimensional Euclidean geometry as a scientific theory that explains and predicts a specific set of properties of some of the entities on flat surfaces. These are theoretical entities such as points, vertices, straight lines and curved lines, circles, polygons, and the like, which we describe in terms of properties such as length, straightness, and angle, the relation between properties of an entity, such as between length and angle, and correlations between properties of two or more entities. In this view, the system of axioms, definitions, conjectures and theorems, and the derivation of theorems from axioms and definitions, is ***the axiomatic component of an empirical theory***.

This unorthodox conceptualisation has an important consequence. In our considered opinion and experience as educators across disciplines, geometry in primary schools is best taught as a branch of observational science. It can be taught as an axiomatic system (a branch of mathematics) *after* learners have had the experience of science. Furthermore, as we see it, geometry as an axiomatic system calls for a certain degree of maturity to cope with the non-visual abstraction and axiomatic proof that mathematics requires, and it is only after the transition from pre-teen to teen years that the human mind-brain is ready for it.

We use a storytelling mode here to articulate our understanding of the relation between theory construction in mathematics and in science. What we have to say is at the very heart of *ways of thinking and understanding* in these two academic enterprises.

8.2 Two Communities of Investigators: An Imagined Tale

Suppose you take a cotton thread, dip it in ink, and throw it on a piece of paper. You will probably get a squiggle of the kind in *Fig. 8-1*.

But if you hold the thread at the two ends, stretch it, and hold it against the paper, still stretching it tight, the mark will be as in *Fig. 8-2*.

And if you take a thread with no ink on it, attach one side of it to a drawing pin fixed on a board, attach the other side to a pencil, and with the thread stretched taut, let the pencil make a mark going all the way it can, you will get *Fig. 8-3*.

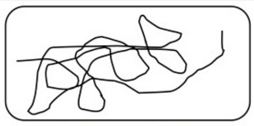

Figure 8-1

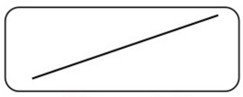

Figure 8-2

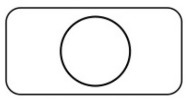

Figure 8-3

Now, if you take two threads, dip each of them in ink of a different colour, and throw them both on a piece of paper, one of them on top of the other, you will see the colours cross each other at several places. But if you make the marks with the two threads stretched tight, you will never see them crossing at two or more distinct places. This is interesting. Why should stretching the threads taut have this effect on their crossing each other?

There are other interesting patterns that you would observe in the behaviour of threads. Suppose you measure the length of the taut thread in *Fig. 8-3*. Call it length A1. Now suppose you place another thread on top of the circular mark in *Fig. 8.3*, matching it exactly, but without stretching it, till the two ends meet. Measure that length. Call it length B1. Now find the ratio A1/B1. If you do this for threads of different lengths A and B, you will find that the ratio A1/B1, A2/B2, A3/B3... is the same. Why should the ratio be constant?

Imagine this. There is a group of investigators who are interested in studying such properties of threads. Let us call it threadology. Threadology focuses on the *length* of threads and their *tension* (stretched taut vs. left free), and the relation between them, ignoring such things as their thickness, colour, breakability, or flammability.

There is another group of investigators who study properties of ropes; call them ropologists. They find that when two ropes are stretched taut, they don't cross each other more than once, exactly as in the case of threads. Furthermore, going through the same processes as the threadologists, the ropologists find that ropes show a difference in length depending on whether or not they are stretched taut. And for different lengths A and B, the ratio A1/B1, A2/B2, A3/B3... is the same.

The threadologists and the ropologists happen to meet. When they talk about their investigation, they are astonished! The general patterns they observe with threads and ropes are the same. Why should this be so?

The two groups happily merge into a single group, and study the properties of both threads and ropes. To see what is common to threads and ropes, they ignore the *differences* between them, and abstract out *what is shared*. This, they call a 'line'. Combining threadology and ropology, they refer to what they study as thropology, and describe their preoccupation as constructing a theory of lines that correctly predicts and explains the properties of lines and their interactions.

Conceptualised this way, thropology is a scientific discipline that seeks to predict and explain certain aspects of our *observations* on threads and ropes. It is not a branch of geometry as a branch of mathematics. How can it be connected to mathematics, and become one of its subdisciplines? We will engage with this question in §8.5.

8.3 Expanding the Scope of Thropology

Our story continues. Unknown to the thropologists, there is another group of investigators who are interested in observing how paper can be folded, and the creases the folds leave. Let us call them creasologists.

Figure 8-4

Suppose you take a sheet of paper and fold it once randomly, running your thumb nail along the *fold* so that it is crisp. Now unfold it. you will see a *crease* on the paper, like the one in *Fig. 8-4*, right?

Now fold it again, and unfold it, such that the paper has another sharp crease. Can you do this in such a way that the two creases cross each other?

Can the creases cross each other at two distinct locations? It won't take you long to discover that the answer is, 'No.' Do you see that this pattern is exactly what the thropologists have discovered in the behaviour of threads and ropes?

The creasologists find another pattern. Suppose you take a sheet of paper P1 and fold it. Then, without unfolding it, you fold it again, such that one part of the first fold lies exactly on the other part. Take another piece of paper P2, and repeat the folding. If you now place the two folded pieces of paper one on top of the other, you find that you can place them in such a way that the two folds of P1 and of P2 align exactly. No matter how many times you do this, no matter what the colour or size of the paper, no matter what the orientation of the folds, you can always align them perfectly.

The creasologists now write a paper describing all their findings, and make it publicly available. When the thropologists read this paper, they are intrigued. That no two creases can cross each other at two distinct places is parallel to what they found in threads and ropes. They look at the other results and find many such patterns that threads, ropes and creases on paper share. They contact the creasologists, and organise a joint meeting to compare notes.

At the end of the meeting, they agree that the behaviour of threads and ropes when stretched taut and of creases left on paper by folding are manifestations of the same abstract pattern.

They also realise that thropologists and creasologists are pursuing the same set of questions, although with different objects: threads and ropes vs. paper; and that it is important for them to unify the knowledge they arrive at.

They decide to use the term ***straight line*** to refer to the abstract concept shared by stretched threads, stretched ropes, and folds and creases on paper; and ***non-straight line*** for the threads and ropes that are not stretched. They now have two kinds of lines, straight and non-straight.

They then discover that if you place the tip of a pen or pencil on a piece of paper and drag it, the result would be a line, either straight or non-straight. If you drag it along the edge of a ruler, it is always a straight line. But if you drag it along the edge of your fingers, you get a non-straight line. And if you drag it along the edge of a disk, the result has the same properties as in *Fig. 8-3*. The thropologists see that the scope of ***thropology*** includes the study of not only threads and ropes, but also folds and creases, and the marks made on paper by dragging a pen along the edge of an object placed on the paper.

By now, there is a large community of investigators exploring the properties of lines using threads, ropes, paper folds and creases, and the edges of objects. Those who have been working on folding paper discover that they can create figures with three straight lines, which they call ***trilaterals***; with four straight lines, which they call ***quadrilaterals***; with five straight lines, which they call ***pentalaterals***; and so on.

Their inquiry has become quite sophisticated by now. They classify the entities they are investigating along different dimensions as follows:

Categories	***line*** ***polylateral*** (entity composed of three or more straight lines)
Sub-categories:	
of ***lines***	***straight*** and ***non-straight***
of ***non-straight lines***	***closed*** (returning to starting point) and ***open***
of ***non-straight lines***	***self-crossing*** (a line crosses itself) and ***non-self-crossing***
of ***polylaterals***	***trilaterals, quadrilaterals, pentalaterals, …***

The investigators now come up with the idea of what they call ***angles***. Take a straight line AB. Now rotate AB around A such that B moves to B', and AB' is a straight line. They refer to the degree of rotation from AB to AB' as an angle. And the joint between two straight lines forming an angle, they refer to as a ***vertex***.

They conceptualise further: If you rotate AB all the way such that B returns to itself, it is called a full rotation. If the rotation is one fourth of a full rotation, it is called a ***right angle***. If it is half a full rotation, it is called a ***straight angle***: *straight* because when BA and AB' are joined together, they make a straight line BAB'.

Then they find that the number of sides that a polylateral has is the same as the number of angles in it. If a polylateral has N sides, it has N angles, and vice versa. So an alternative name for polylaterals is ***polyangles***.

To connect this to terminology that you are familiar with, X-angles are called X-*gons* in Latin, so they came to be called ***polygons***. But trilaterals got the label ***triangles***, pentalaterals got called ***pentagons,*** hexalaterals are ***hexagons,*** and so on. And in this mixed-up terminology, quadrilaterals remained ***quadrilaterals***.

To continue with some familiar terminology, polygons in which all the angles are equal are called ***equiangular*** polygons, and those in which all the lines (called 'sides') were equal are called ***equilateral*** polygons. And polygons that are both equilateral and equiangular are called ***regular*** polygons.

By this time, the investigators have a large number of what they call ***observational generalisations.*** These are statements that they consider to be true of all the members of a category (a population), or about the relation between members of two or more categories. For instance, the 'angle-sum hypothesis' is an observational generalisation: it says that the sum of angles of a triangle is a straight angle. Another observational generalisation says that an equiangular triangle is also equilateral, and vice versa. Yet another observational generalisation says that for every triangle, there exists a circle that passes through every vertex of the triangle; and there exists a circle that touches every side of the triangle without crossing it.

The most famous observational generalisation is called the Pythagoras Generalisation; we will reserve that story for another time.

8.4 Knowledge vs. Knowledge Claims: the Issue of Proof

In the process of their explorations, some of the investigators become sensitive to the distinction between knowledge and knowledge claims. A ***knowledge claim*** is a statement that we believe to be true, but has not yet been established as part of knowledge. To be admitted as ***knowledge*** in an academic field, the claim has be proved.

So the investigators turn their attention to the challenging task of proving the knowledge claims they have arrived at. The first such claim is the statement:

> No two straight lines can meet or intersect (cross) at two distinct points.

Creasologists working on paper folding set up a research project in which they ask many researchers to fold a piece of paper twice such that the resulting creases cross at two distinct points. After doing this with a sample of more than 10,000 pieces of paper, they do not find a single pair of creases that cross at two distinct points.

They offer the following proof:

> ***Conjecture**:* No two straight lines can meet or cross at two distinct points.
> ***Reasoning**:* *Sample to Population*
> ***Proof:***
>
> We have observed a large sample (more than 10,000) of pieces of paper with two creases on each piece.
>
> We have not found a single piece of paper in which two creases meet or cross at two distinct points.
>
> Therefore, in the absence of evidence to the contrary, we conclude that no two creases can meet or cross at two distinct points.

Can this be taken as a proof for the claim that in the entire population of straight lines, no two straight lines can meet or cross at two distinct points? No. All that the proof above shows is that in a population of creases on paper, no two creases can meet or cross at two distinct points. What about the population of stretched threads, or the population of stretched ropes, or tips of pens dragged across a straight edge on paper? All of them count as straight lines. So the sample may be very ***large***, but it is not ***representative*** of the population of straight lines.

To solve this problem of representativeness, those who work on threads, ropes and pen marks on paper also examine large samples. None of them finds a single case where two straight lines meet or cross at two distinct points. So the jury of investigators who examine the knowledge claim (the conjecture in the box above) decides to treat the claim as having been proved, and hence to accept it as part of knowledge.

The rope researchers come up with a similar proof for the claim that the sum of angles in a triangle is equal to a straight angle. For this, they decide to measure the angles. To measure angles, they use the concept of ***degree***. They divide a full rotation into 360 equal parts, with each part being a degree, so that a full rotation is 360 degrees. They use an instrument called a 'protractor' that is calibrated such that a straight angle is 180 degrees. A protractor allows them to count the number of degree marks on it to measure angles.

To construct a rope triangle, they need six people: one to hold each side of a rope tightly stretched, with three such ropes. And a seventh person measures the three angles with a protractor. They measure the angles of a large number of such triangles, and add together the degrees of the three angles of each triangle. When they add up the degrees of all the triangles, and find the mean by dividing the total number by the number of triangles, they find the mean to be 179 degrees, plus or minus two degrees.

Using this idea, they offer the following proof:

Conjecture: The sum of angles in a triangle is 180 degrees.
Reasoning: *Sample to Population*
Proof:

We have observed a large sample (more than 10,000) of rope triangles and measured their angles.

The mean of the sum of the angles in a triangle is 179 degrees, plus or minus two degrees.

We have not found a single triangle in which the sum is less than 177 degrees or more than 181 degrees.

Therefore, in the absence of evidence to the contrary, we conclude that the sum of angles in a triangle not less than 177 degrees, and not more than 181 degrees.

The creasologists come up with similar proofs for a large number of their knowledge claims to establish them as knowledge.

8.5 Geometry as an Axiomatic System

The proofs we have seen use Sample to Population Reasoning, which uses what is called *inductive logic*, and is based on observational reports on a large representative sample. This kind of proof is called ***Empirical Proof***.

To continue our story, at this time there is a graduate student called Oclid who has been studying *deductive logic*. She suggests a different way of proving knowledge claims. She calls her proofs ***Axiomatic Proofs***.

An ***axiomatic system*** is a configuration of *axioms, definitions, their logical consequences, and the derivation of those consequences*. What is called a 'proof' in an axiomatic system is the demonstration that the alleged logical consequence (called ***conjecture***) of the axioms and definitions is a true logical consequence (called a ***theorem***), by demonstrating that it can be derived from the axioms and definitions of the system.

So while the premises of an ***empirical system*** are observational reports, those of an axiomatic system are ***axioms*** and ***definitions***.

In an axiomatic system, the only form of logic that can be used for establishing truth claims is classical deductive logic. An empirical system, however, allows for the use of different systems of logic, including sample to population reasoning, classical deductive logic, and so on, for establishing a claim.

Oclid's way is to begin by defining the concepts found in thropology and creasology. She begins with a description of the entity she wants to call line:

1. *Line:* an entity that exists in a space. It has length, but not breadth or thickness.

This is a description: it does not qualify as a definition. But using this undefined concept, she proceeds to define the other concepts she needs:

2. *Point:* an entity that exist at the ends of a line, with no length, breadth or thickness, and is what is shared by two lines that meet or cross.
3. *Straight line*: the shortest path between two distinct points.
4. *Polygon:* an entity made up of straight lines and only straight lines with as many vertices as there are straight lines.
5. *Triangle:* A three-sided polygon

6. *Quadrilateral:* A four-sided polygon
7. *Parallelogram:* A quadrilateral in which the opposite sides are parallel
8. *Rectangle:* An equiangular parallelogram
9. *Square:* An equilateral rectangle
10. *Circle*: A closed line in which every point is equidistant from a central point.

She also proposes a few axioms, some of which are specific to geometry (A-C), and others that are more general (D-E):

A. *Unique Path:* For any two distinct points A and B, there exists one and only one shortest path between them.

B. *Non-colinearity*: In a polygon, no three vertices can be colinear. (Co-linear: existing in a single straight line)

C. *Openness of straight lines:* No straight line, however extended on both sides, can meet itself.

D. *Acceptance of Logical Consequences:* If we accept a set of premise propositions as true, we must also accept as true those propositions that are their logical consequences.

E. *Rejection of Logical Contradictions:* We must reject combinations of logically contradictory propositions as false.

Oclid knows that some of the definitions contain concepts which themselves call for a definition. For example, in the definition of 'point', what is the concept of length, also implicit in the reference to the *shortest* path in the definition of 'straight line'?

She is also aware that concepts like *equiangular* and *equilateral* need to be defined.

Finally, she is aware that the definitions she has come up with may lack adequate clarity, and may even have inconsistencies that she has not been able to detect. But the definitions given above in (1)-(10) and the axioms in A-E serve as a good starting point for constructing an axiomatic theory of lines.

Recall that a ***proof*** is a set of statements composed of (a) premises, (b) conclusions, and (c) steps of reasoning (called derivation) from the premises to the conclusion (the PDC structure you have come across in earlier units), to demonstrate that the conclusions are the logical consequences of the premises. In Axiomatic Proofs, the premises are definitions and axioms.

To get a sense of how Axiomatic Proofs work, let us take as an example Oclid's proof for the claim that no two straight lines can meet or cross at two distinct points.

An Axiomatic Proof:

To prove: No two straight lines can meet or cross at two distinct points.

Premises:

P1: A straight line is the shortest path between two distinct points. (DEF 3)

P2: *Unique Path*: For any two distinct points A and B, there exists one and only one shortest path between them. (AXIOM A)

P3: *Rejection of Logical Contradictions*: We must reject combinations of logically contradictory propositions as false. (AXIOM E)

Derivation: S(teps of reasoning)

S1: Assume that there exist two straight lines that meet/cross at two distinct points A and B [contrary to what we need to prove].

S2: By P1and S1, there are two shortest paths between A and B.

S3: By S2, there are two shortest paths between A and B, and by P2, there cannot be two shortest paths between A and B.

S4: S3 contains a logical contradiction, and hence by P3, S3 is false.

S5: To eliminate what is false, we must reject at least one of the premises that lead to the contradiction.

S6: We choose to reject S1.

S7: Hence, we conclude that no two straight lines can meet/cross at two distinct points.

Conclusion: No two straight lines can meet/cross at two distinct points. (QED)

To take another example, let us return to the angle sum claim, proved as an observational generalisation in the previous section. Let us now prove it axiomatically.

An Axiomatic Proof:

To prove: The sum of angles in a triangle is equal to two right angles (= a straight angle).

Premises:

P1: When a straight line crosses two parallel lines, their internal alternative angles are congruent (i.e., their magnitudes are equal).

Derivation:

S1: In your mind, construct any arbitrary triangle ABC. It can be as in any of the triangles in *Fig. 8-5*:

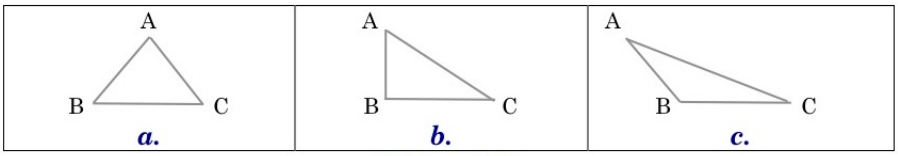

Figure 8-5

S2: Again in your mind, draw a straight line DE through A, such that DE is parallel to BC, as in *Fig. 8-6*:

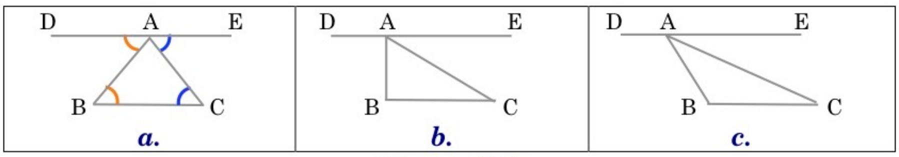

Figure 8-6

S3: By P1, ∠ DAB = ∠ ABC (marked))

∠ EAC = ∠ ACB (marked))

The sum of angles of triangle ABC

= ∠ ABC + ∠ ACB + ∠ BAC

= ∠ DAB + ∠ EAC + ∠ BAC

= a straight angle = two right angles

The same argument applies to the remaining figures.

Conclusion: The sum of angles in a triangle is equal to a straight angle. (QED)

8.6 Alternative Definitions and Axioms

As mentioned above, Oclid conceptualises lines as entities with length but no breadth. What that means is, no matter how many lines we place side by side, the combination will not become a rectangle: adding infinitely many zeros doesn't give you a number more than zero. Similarly, adding points (entities of zero size) won't give you a line.

The essence of measurement in science is that of dividing a quantity into equal parts, and counting the numbers of parts needed to do the measuring. Take, for instance, the length that we call a kilometer. If we divide that length into a thousand equal parts, each part is a meter. We can now 'measure' the distance from a school gate to the next intersection by counting the number of meters from the school gate to the intersection (and ignoring the lengths which are less than a meter). When measuring the height of a person, we divide a meter into a hundred equal parts called centimeters, and count the number of centimeters from the bottom of a person's feet to the top of her head. When measuring the thickness of a metal wire, we divide a centimeter into ten equal parts called millimeters and count the millimeters. If we want still greater precision, we can divide millimeters too. And if we divide a meter into a billion parts, we get a nanometer. But no matter how small, a unit of measurement cannot be zero.

Now, instead of conceptualising a line as having zero breadth, what if we conceptualise it as follows:

> A line is an entity which has length and whose breadth is greater than zero, but less than what can be measured.

As far as current physics is concerned, the smallest measurable length is known as Planck's length: 1.6 x 10^{-35} meters (See https://www.fnal.gov/pub/today/archive/archive_2013/today13-11-01_NutshellReadMore.html) So let us reformulate our description as follows:

> A line is an entity which has length, and whose breadth is greater than zero, and less than Planck's length.

We now define 'point' as Oclid did, as an entity shared by two lines that intersect or meet. We can now add the following premises to our description of lines:

> A line is composed of points such that every point has exactly two neighbours, except the end points, which have exactly one neighbour each.

Notice that Oclid has no definition of length, and hence no way of determining whether an irregularly curved line is longer or shorter than a straight line. The conception of lines being composed of points as specified above solves that problem.

What we have outlined above is an alternative conceptualisation of points and lines. Does this conceptualisation yield the same theory of geometry as Oclid's theory? Or does it yield a different theory? We leave that for you to ponder over and figure out.

8.7 Final Remarks

We are talking about Euclid's theory of two-dimensional geometry What is this theory about?

With the hindsight of the multiple geometries that began to appear in the second half of the nineteenth century, and continued to appear in the twentieth century, one may say that:

> Mathematics is about logically consistent imagined worlds; and Euclidean geometry is about one of those worlds.

If the properties of the logically consistent imagined world investigated by Euclidean geometry correspond closely to the properties of the actual world we live in (the world that we experience), then we may say that Euclidean geometry is a good model of the world we live in.

In Euclid's conception of geometry, he saw the axioms of the theory as self-evident truths. And he took them as truths about objects in a flat two-dimensional space in our world of experience.

The axioms were the equivalent of the scientific laws of that world. He took for granted that the theorems of that theory (the predictions) matched the observational generalisations on the objects in that world. Hence, what he proposed was an axiomatic system about the world that he experienced. The system correctly predicted and explained the world of his experience. So this was an axiomatic system of empirical knowledge. This was a conceptualisation that worked well, for instance, for the purposes of physics.

With that remark, we return to what we said at the beginning of this unit, which the readers would be in a better position to understand now. In the preceding sections, we explored two-dimensional Euclidean geometry first as an observational science, and then as an axiomatic system in mathematics. That exploration led to a somewhat unconventional conceptualisation of two-dimensional Euclidean geometry as a scientific theory that explains and predicts a specific set of properties of entities on flat surfaces. The entities, such as points, lines (straight and curved), vertices, angles, circles, polygons, and the like were abstractions of entities observed in the world of experience: threads, ropes, creases on pieces of paper, and so on. And we describe these entities in terms of properties such as length, straightness and angle; the relation between the properties of an entity, such as between length and angle; and correlations between properties of two or more entities.

In this view, the axiomatic system of axioms, definitions, conjectures, theorems, and the derivation of theorems from axioms and definitions is *the axiomatic component of an empirical theory.*

To repeat what we said in section 1, an important consequence of this unconventional conceptualisation is what it tells us about the optimal the design of school curricula. We would like to suggest that it would be best to introduce Euclidean geometry in primary schools as a branch of observational science. Learners can be introduced geometry as an axiomatic system — a branch of mathematics — after they have had the experience of science, which is about the world around them.

As pointed out in §1, learners need a certain degree of intellectual maturity in order to cope with the non-visual abstraction and axiomatic proof that mathematics, and hence geometry, calls for. We believe that learners are ready to engage with such abstract axiomatic systems only after the transition from pre-teen to teen years.

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