UNIT 8: GEOMETRY AS SCIENCE AND AS MATH

8.1 Introductory Remarks

This unit, we hope, will provide an experiential understanding of the distinction between mathematical and scientific theories. To this end, we explore two-dimensional Euclidean Geometry, first as an observational science, and then as an axiomatic system in mathematics. That exploration leads to a somewhat unconventional conceptualisation of two-dimensional Euclidean geometry as a scientific theory that explains and predicts a specific set of properties of some of the entities on flat surfaces. These are theoretical entities such as points, vertices, straight lines and curved lines, circles, polygons, and the like, which we describe in terms of properties such as length, straightness, and angle, the relation between properties of an entity, such as between length and angle, and correlations between properties of two or more entities. In this view, the system of axioms, definitions, conjectures and theorems, and the derivation of theorems from axioms and definitions, is ***the axiomatic component of an empirical theory***.

This unorthodox conceptualisation has an important consequence. In our considered opinion and experience as educators across disciplines, geometry in primary schools is best taught as a branch of observational science. It can be taught as an axiomatic system (a branch of mathematics) *after* learners have had the experience of science. Furthermore, as we see it, geometry as an axiomatic system calls for a certain degree of maturity to cope with the non-visual abstraction and axiomatic proof that mathematics requires, and it is only after the transition from pre-teen to teen years that the human mind-brain is ready for it.

We use a storytelling mode here to articulate our understanding of the relation between theory construction in mathematics and in science. What we have to say is at the very heart of *ways of thinking and understanding* in these two academic enterprises.

8.2 Two Communities of Investigators: An Imagined Tale

Suppose you take a cotton thread, dip it in ink, and throw it on a piece of paper. You will probably get a squiggle of the kind in *Fig. 8-1*.

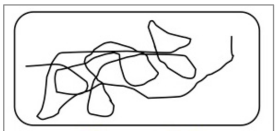

Figure 8-1

But if you hold the thread at the two ends, stretch it, and hold it against the paper, still stretching it tight, the mark will be as in *Fig. 8-2*.

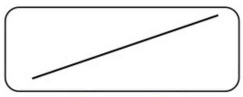

Figure 8-2

And if you take a thread with no ink on it, attach one side of it to a drawing pin fixed on a board, attach the other side to a pencil, and with the thread stretched taut, let the pencil make a mark going all the way it can, you will get *Fig. 8-3*.

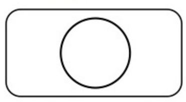

Figure 8-3

Now, if you take two threads, dip each of them in ink of a different colour, and throw them both on a piece of paper, one of them on top of the other, you will see the colours cross each other at several places. But if you make the marks with the two threads stretched tight, you will never see them crossing at two or more distinct places. This is interesting. Why should stretching the threads taut have this effect on their crossing each other?

There are other interesting patterns that you would observe in the behaviour of threads. Suppose you measure the length of the taut thread in *Fig. 8-3*. Call it length A1. Now suppose you place another thread on top of the circular mark in *Fig. 8.3*, matching it exactly, but without stretching it, till the two ends meet. Measure that length. Call it length B1. Now find the ratio A1/B1. If you do this for threads of different lengths A and B, you will find that the ratio A1/B1, A2/B2, A3/B3... is the same. Why should the ratio be constant?

Imagine this. There is a group of investigators who are interested in studying such properties of threads. Let us call it threadology. Threadology focuses on the *length* of threads and their *tension* (stretched taut vs. left free), and the relation between them, ignoring such things as their thickness, colour, breakability, or flammability.

There is another group of investigators who study properties of ropes; call them ropologists. They find that when two ropes are stretched taut, they don't cross each other more than once, exactly as in the case of threads. Furthermore, going through the same processes as the threadologists, the ropologists find that ropes show a difference in length depending on whether or not they are stretched taut. And for different lengths A and B, the ratio A1/B1, A2/B2, A3/B3... is the same.

The threadologists and the ropologists happen to meet. When they talk about their investigation, they are astonished! The general patterns they observe with threads and ropes are the same. Why should this be so?

The two groups happily merge into a single group, and study the properties of both threads and ropes. To see what is common to threads and ropes, they ignore the *differences* between them, and abstract out *what is shared*. This, they call a 'line'. Combining threadology and ropology, they refer to what they study as thropology, and describe their preoccupation as constructing a theory of lines that correctly predicts and explains the properties of lines and their interactions.

Conceptualised this way, thropology is a scientific discipline that seeks to predict and explain certain aspects of our *observations* on threads and ropes. It is not a branch of geometry as a branch of mathematics. How can it be connected to mathematics, and become one of its subdisciplines? We will engage with this question in §8.5.

8.3 Expanding the Scope of Thropology

Our story continues. Unknown to the thropologists, there is another group of investigators who are interested in observing how paper can be folded, and the creases the folds leave. Let us call them creasologists.

Suppose you take a sheet of paper and fold it once randomly, running your thumb nail along the *fold* so that it is crisp. Now unfold it. you will see a *crease* on the paper, like the one in *Fig. 8-4*, right?

Figure 8-4

Now fold it again, and unfold it, such that the paper has another sharp crease. Can you do this in such a way that the two creases cross each other?

Can the creases cross each other at two distinct locations? It won't take you long to discover that the answer is, 'No.' Do you see that this pattern is exactly what the thropologists have discovered in the behaviour of threads and ropes?

The creasologists find another pattern. Suppose you take a sheet of paper P1 and fold it. Then, without unfolding it, you fold it again, such that one part of the first fold lies exactly on the other part. Take another piece of paper P2, and repeat the folding. If you now place the two folded pieces of paper one on top of the other, you find that you can place them in such a way that the two folds of P1 and of P2 align exactly. No matter how many times you do this, no matter what the colour or size of the paper, no matter what the orientation of the folds, you can always align them perfectly.

The creasologists now write a paper describing all their findings, and make it publicly available. When the thropologists read this paper, they are intrigued. That no two creases can cross each other at two distinct places is parallel to what they found in threads and ropes. They look at the other results and find many such patterns that threads, ropes and creases on paper share. They contact the creasologists, and organise a joint meeting to compare notes.

At the end of the meeting, they agree that the behaviour of threads and ropes when stretched taut and of creases left on paper by folding are manifestations of the same abstract pattern.

They also realise that thropologists and creasologists are pursuing the same set of questions, although with different objects: threads and ropes vs. paper; and that it is important for them to unify the knowledge they arrive at.

They decide to use the term ***straight line*** to refer to the abstract concept shared by stretched threads, stretched ropes, and folds and creases on paper; and ***non-straight line*** for the threads and ropes that are not stretched. They now have two kinds of lines, straight and non-straight.

They then discover that if you place the tip of a pen or pencil on a piece of paper and drag it, the result would be a line, either straight or non-straight. If you drag it along the edge of a ruler, it is always a straight line. But if you drag it along the edge of your fingers, you get a non-straight line. And if you drag it along the edge of a disk, the result has the same properties as in *Fig. 8-3*. The thropologists see that the scope of ***thropology*** includes the study of not only threads and ropes, but also folds and creases, and the marks made on paper by dragging a pen along the edge of an object placed on the paper.

By now, there is a large community of investigators exploring the properties of lines using threads, ropes, paper folds and creases, and the edges of objects. Those who have been working on folding paper discover that they can create figures with three straight lines, which they call ***trilaterals***; with four straight lines, which they call ***quadrilaterals***; with five straight lines, which they call ***pentalaterals***; and so on.

Their inquiry has become quite sophisticated by now. They classify the entities they are investigating along different dimensions as follows:

Categories	***line*** ***polylateral*** (entity composed of three or more straight lines)
Sub-categories:	
of ***lines***	***straight*** and ***non-straight***
of ***non-straight lines***	***closed*** (returning to starting point) and ***open***
of ***non-straight lines***	***self-crossing*** (a line crosses itself) and ***non-self-crossing***
of ***polylaterals***	***trilaterals, quadrilaterals, pentalaterals, …***

The investigators now come up with the idea of what they call ***angles***. Take a straight line AB. Now rotate AB around A such that B moves to B', and AB' is a straight line. They refer to the degree of rotation from AB to AB' as an angle. And the joint between two straight lines forming an angle, they refer to as a ***vertex***.

They conceptualise further: If you rotate AB all the way such that B returns to itself, it is called a full rotation. If the rotation is one fourth of a full rotation, it is called a ***right angle***. If it is half a full rotation, it is called a ***straight angle***: *straight* because when BA and AB' are joined together, they make a straight line BAB'.

Then they find that the number of sides that a polylateral has is the same as the number of angles in it. If a polylateral has N sides, it has N angles, and vice versa. So an alternative name for polylaterals is ***polyangles***.

To connect this to terminology that you are familiar with, X-angles are called X-*gons* in Latin, so they came to be called ***polygons***. But trilaterals got the label ***triangles***, pentalaterals got called ***pentagons,*** hexalaterals are ***hexagons,*** and so on. And in this mixed-up terminology, quadrilaterals remained ***quadrilaterals***.

To continue with some familiar terminology, polygons in which all the angles are equal are called ***equiangular*** polygons, and those in which all the lines (called 'sides') were equal are called ***equilateral*** polygons. And polygons that are both equilateral and equiangular are called ***regular*** polygons.

By this time, the investigators have a large number of what they call ***observational generalisations.*** These are statements that they consider to be true of all the members of a category (a population), or about the relation between members of two or more categories. For instance, the 'angle-sum hypothesis' is an observational generalisation: it says that the sum of angles of a triangle is a straight angle. Another observational generalisation says that an equiangular triangle is also equilateral, and vice versa. Yet another observational generalisation says that for every triangle, there exists a circle that passes through every vertex of the triangle; and there exists a circle that touches every side of the triangle without crossing it.

The most famous observational generalisation is called the Pythagoras Generalisation; we will reserve that story for another time.

8.4 Knowledge vs. Knowledge Claims: the Issue of Proof

In the process of their explorations, some of the investigators become sensitive to the distinction between knowledge and knowledge claims. A ***knowledge claim*** is a statement that we believe to be true, but has not yet been established as part of knowledge. To be admitted as ***knowledge*** in an academic field, the claim has be proved.

So the investigators turn their attention to the challenging task of proving the knowledge claims they have arrived at. The first such claim is the statement:

> No two straight lines can meet or intersect (cross) at two distinct points.

Creasologists working on paper folding set up a research project in which they ask many researchers to fold a piece of paper twice such that the resulting creases cross at two distinct points. After doing this with a sample of more than 10,000 pieces of paper, they do not find a single pair of creases that cross at two distinct points.

They offer the following proof:

***Conjecture**:* No two straight lines can meet or cross at two distinct points.
***Reasoning**:* *Sample to Population*
***Proof**:*

We have observed a large sample (more than 10,000) of pieces of paper with two creases on each piece.

We have not found a single piece of paper in which two creases meet or cross at two distinct points.

Therefore, in the absence of evidence to the contrary, we conclude that no two creases can meet or cross at two distinct points.

Can this be taken as a proof for the claim that in the entire population of straight lines, no two straight lines can meet or cross at two distinct points? No. All that the proof above shows is that in a population of creases on paper, no two creases can meet or cross at two distinct points. What about the population of stretched threads, or the population of stretched ropes, or tips of pens dragged across a straight edge on paper? All of them count as straight lines. So the sample may be very ***large***, but it is not ***representative*** of the population of straight lines.

To solve this problem of representativeness, those who work on threads, ropes and pen marks on paper also examine large samples. None of them finds a single case where two straight lines meet or cross at two distinct points. So the jury of investigators who examine the knowledge claim (the conjecture in the box above) decides to treat the claim as having been proved, and hence to accept it as part of knowledge.

The rope researchers come up with a similar proof for the claim that the sum of angles in a triangle is equal to a straight angle. For this, they decide to measure the angles. To measure angles, they use the concept of ***degree***. They divide a full rotation into 360 equal parts, with each part being a degree, so that a full rotation is 360 degrees. They use an instrument called a 'protractor' that is calibrated such that a straight angle is 180 degrees. A protractor allows them to count the number of degree marks on it to measure angles.

To construct a rope triangle, they need six people: one to hold each side of a rope tightly stretched, with three such ropes. And a seventh person measures the three angles with a protractor. They measure the angles of a large number of such triangles, and add together the degrees of the three angles of each triangle. When they add up the degrees of all the triangles, and find the mean by dividing the total number by the number of triangles, they find the mean to be 179 degrees, plus or minus two degrees.

Using this idea, they offer the following proof:

Conjecture: The sum of angles in a triangle is 180 degrees.
Reasoning: *Sample to Population*
Proof:

We have observed a large sample (more than 10,000) of rope triangles and measured their angles.

The mean of the sum of the angles in a triangle is 179 degrees, plus or minus two degrees.

We have not found a single triangle in which the sum is less than 177 degrees or more than 181 degrees.

Therefore, in the absence of evidence to the contrary, we conclude that the sum of angles in a triangle not less than 177 degrees, and not more than 181 degrees.

The creasologists come up with similar proofs for a large number of their knowledge claims to establish them as knowledge.

8.5 Geometry as an Axiomatic System

The proofs we have seen use Sample to Population Reasoning, which uses what is called *inductive logic*, and is based on observational reports on a large representative sample. This kind of proof is called ***Empirical Proof***.

To continue our story, at this time there is a graduate student called Oclid who has been studying *deductive logic*. She suggests a different way of proving knowledge claims. She calls her proofs ***Axiomatic Proofs***.

An ***axiomatic system*** is a configuration of *axioms, definitions, their logical consequences, and the derivation of those consequences*. What is called a 'proof' in an axiomatic system is the demonstration that the alleged logical consequence (called ***conjecture***) of the axioms and definitions is a true logical consequence (called a ***theorem***), by demonstrating that it can be derived from the axioms and definitions of the system.

So while the premises of an ***empirical system*** are observational reports, those of an axiomatic system are ***axioms*** and ***definitions***.

In an axiomatic system, the only form of logic that can be used for establishing truth claims is classical deductive logic. An empirical system, however, allows for the use of different systems of logic, including sample to population reasoning, classical deductive logic, and so on, for establishing a claim.

Oclid's way is to begin by defining the concepts found in thropology and creasology. She begins with a description of the entity she wants to call line:

1. *Line:* an entity that exists in a space. It has length, but not breadth or thickness.

This is a description: it does not qualify as a definition. But using this undefined concept, she proceeds to define the other concepts she needs:

2. *Point:* an entity that exist at the ends of a line, with no length, breadth or thickness, and is what is shared by two lines that meet or cross.
3. *Straight line*: the shortest path between two distinct points.
4. *Polygon:* an entity made up of straight lines and only straight lines with as many vertices as there are straight lines.
5. *Triangle:* A three-sided polygon

6. *Quadrilateral:* A four-sided polygon
7. *Parallelogram:* A quadrilateral in which the opposite sides are parallel
8. *Rectangle:* An equiangular parallelogram
9. *Square:* An equilateral rectangle
10. *Circle*: A closed line in which every point is equidistant from a central point.

She also proposes a few axioms, some of which are specific to geometry (A-C), and others that are more general (D-E):

A. *Unique Path:* For any two distinct points A and B, there exists one and only one shortest path between them.

B. *Non-colinearity*: In a polygon, no three vertices can be colinear.
(Co-linear: existing in a single straight line)

C. *Openness of straight lines:* No straight line, however extended on both sides, can meet itself.

D. *Acceptance of Logical Consequences:* If we accept a set of premise propositions as true, we must also accept as true those propositions that are their logical consequences.

E. *Rejection of Logical Contradictions:* We must reject combinations of logically contradictory propositions as false.

Oclid knows that some of the definitions contain concepts which themselves call for a definition. For example, in the definition of 'point', what is the concept of length, also implicit in the reference to the *shortest* path in the definition of 'straight line'?

She is also aware that concepts like *equiangular* and *equilateral* need to be defined.

Finally, she is aware that the definitions she has come up with may lack adequate clarity, and may even have inconsistencies that she has not been able to detect. But the definitions given above in (1)-(10) and the axioms in A-E serve as a good starting point for constructing an axiomatic theory of lines.

Recall that a ***proof*** is a set of statements composed of (a) premises, (b) conclusions, and (c) steps of reasoning (called derivation) from the premises to the conclusion (the PDC structure you have come across in earlier units), to demonstrate that the conclusions are the logical consequences of the premises. In Axiomatic Proofs, the premises are definitions and axioms.

To get a sense of how Axiomatic Proofs work, let us take as an example Oclid's proof for the claim that no two straight lines can meet or cross at two distinct points.

An Axiomatic Proof:

To prove: No two straight lines can meet or cross at two distinct points.

Premises:

P1: A straight line is the shortest path between two distinct points. (DEF 3)

P2: *Unique Path*: For any two distinct points A and B, there exists one and only one shortest path between them. (AXIOM A)

P3: *Rejection of Logical Contradictions*: We must reject combinations of logically contradictory propositions as false. (AXIOM E)

Derivation: S(teps of reasoning)

S1: Assume that there exist two straight lines that meet/cross at two distinct points A and B [contrary to what we need to prove].

S2: By P1and S1, there are two shortest paths between A and B.

S3: By S2, there are two shortest paths between A and B, and by P2, there cannot be two shortest paths between A and B.

S4: S3 contains a logical contradiction, and hence by P3, S3 is false.

S5: To eliminate what is false, we must reject at least one of the premises that lead to the contradiction.

S6: We choose to reject S1.

S7: Hence, we conclude that no two straight lines can meet/cross at two distinct points.

Conclusion: No two straight lines can meet/cross at two distinct points. (QED)

To take another example, let us return to the angle sum claim, proved as an observational generalisation in the previous section. Let us now prove it axiomatically.

An Axiomatic Proof:

To prove: The sum of angles in a triangle is equal to two right angles (= a straight angle).

Premises:

P1: When a straight line crosses two parallel lines, their internal alternative angles are congruent (i.e., their magnitudes are equal).

Derivation:

S1: In your mind, construct any arbitrary triangle ABC. It can be as in any of the triangles in *Fig. 8-5*:

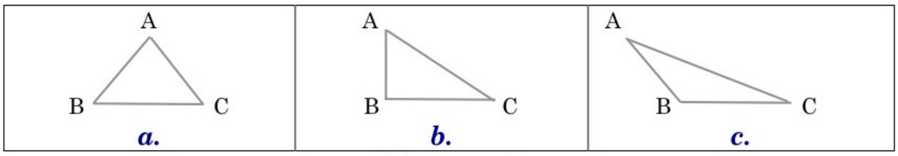

Figure 8-5

S2: Again in your mind, draw a straight line DE through A, such that DE is parallel to BC, as in *Fig. 8-6*:

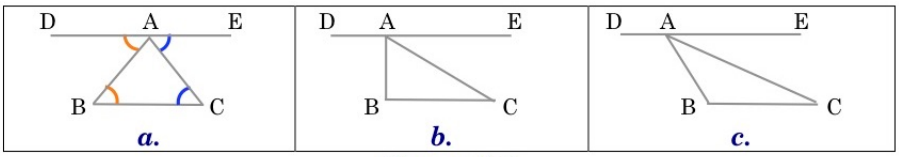

Figure 8-6

S3: By P1, ∠ DAB = ∠ ABC (marked))
∠ EAC = ∠ ACB (marked))
The sum of angles of triangle ABC
= ∠ ABC + ∠ ACB + ∠ BAC
= ∠ DAB + ∠ EAC + ∠ BAC
= a straight angle = two right angles

The same argument applies to the remaining figures.

Conclusion: The sum of angles in a triangle is equal to a straight angle. (QED)

8.6 Alternative Definitions and Axioms

As mentioned above, Oclid conceptualises lines as entities with length but no breadth. What that means is, no matter how many lines we place side by side, the combination will not become a rectangle: adding infinitely many zeros doesn't give you a number more than zero. Similarly, adding points (entities of zero size) won't give you a line.

The essence of measurement in science is that of dividing a quantity into equal parts, and counting the numbers of parts needed to do the measuring. Take, for instance, the length that we call a kilometer. If we divide that length into a thousand equal parts, each part is a meter. We can now 'measure' the distance from a school gate to the next intersection by counting the number of meters from the school gate to the intersection (and ignoring the lengths which are less than a meter). When measuring the height of a person, we divide a meter into a hundred equal parts called centimeters, and count the number of centimeters from the bottom of a person's feet to the top of her head. When measuring the thickness of a metal wire, we divide a centimeter into ten equal parts called millimeters and count the millimeters. If we want still greater precision, we can divide millimeters too. And if we divide a meter into a billion parts, we get a nanometer. But no matter how small, a unit of measurement cannot be zero.

Now, instead of conceptualising a line as having zero breadth, what if we conceptualise it as follows:

> A line is an entity which has length and whose breadth is greater than zero, but less than what can be measured.

As far as current physics is concerned, the smallest measurable length is known as Planck's length: 1.6×10^{-35} meters (See https://www.fnal.gov/pub/today/archive/archive_2013/today13-11-01_NutshellReadMore.html) So let us reformulate our description as follows:

> A line is an entity which has length, and whose breadth is greater than zero, and less than Planck's length.

We now define 'point' as Oclid did, as an entity shared by two lines that intersect or meet. We can now add the following premises to our description of lines:

> A line is composed of points such that every point has exactly two neighbours, except the end points, which have exactly one neighbour each.

Notice that Oclid has no definition of length, and hence no way of determining whether an irregularly curved line is longer or shorter than a straight line. The conception of lines being composed of points as specified above solves that problem.

What we have outlined above is an alternative conceptualisation of points and lines. Does this conceptualisation yield the same theory of geometry as Oclid's theory? Or does it yield a different theory? We leave that for you to ponder over and figure out.

8.7 Final Remarks

We are talking about Euclid's theory of two-dimensional geometry What is this theory about?

With the hindsight of the multiple geometries that began to appear in the second half of the nineteenth century, and continued to appear in the twentieth century, one may say that:

> Mathematics is about logically consistent imagined worlds; and Euclidean geometry is about one of those worlds.

If the properties of the logically consistent imagined world investigated by Euclidean geometry correspond closely to the properties of the actual world we live in (the world that we experience), then we may say that Euclidean geometry is a good model of the world we live in.

In Euclid's conception of geometry, he saw the axioms of the theory as self-evident truths. And he took them as truths about objects in a flat two-dimensional space in our world of experience.

The axioms were the equivalent of the scientific laws of that world. He took for granted that the theorems of that theory (the predictions) matched the observational generalisations on the objects in that world. Hence, what he proposed was an axiomatic system about the world that he experienced. The system correctly predicted and explained the world of his experience. So this was an axiomatic system of empirical knowledge. This was a conceptualisation that worked well, for instance, for the purposes of physics.

With that remark, we return to what we said at the beginning of this unit, which the readers would be in a better position to understand now. In the preceding sections, we explored two-dimensional Euclidean geometry first as an observational science, and then as an axiomatic system in mathematics. That exploration led to a somewhat unconventional conceptualisation of two-dimensional Euclidean geometry as a scientific theory that explains and predicts a specific set of properties of entities on flat surfaces. The entities, such as points, lines (straight and curved), vertices, angles, circles, polygons, and the like were abstractions of entities observed in the world of experience: threads, ropes, creases on pieces of paper, and so on. And we describe these entities in terms of properties such as length, straightness and angle; the relation between the properties of an entity, such as between length and angle; and correlations between properties of two or more entities.

In this view, the axiomatic system of axioms, definitions, conjectures, theorems, and the derivation of theorems from axioms and definitions is *the axiomatic component of an empirical theory.*

To repeat what we said in section 1, an important consequence of this unconventional conceptualisation is what it tells us about the optimal the design of school curricula. We would like to suggest that it would be best to introduce Euclidean geometry in primary schools as a branch of observational science. Learners can be introduced geometry as an axiomatic system — a branch of mathematics — after they have had the experience of science, which is about the world around them.

As pointed out in §1, learners need a certain degree of intellectual maturity in order to cope with the non-visual abstraction and axiomatic proof that mathematics, and hence geometry, calls for. We believe that learners are ready to engage with such abstract axiomatic systems only after the transition from pre-teen to teen years.

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