UNIT 5: A THEORY OF CIRCLES

5.1 Circles: as Regions vs. Boundaries

Let us go back to the concepts of regions and boundaries introduced earlier. Consider the two 'circles' in Figs. 5-1 and 5-2:

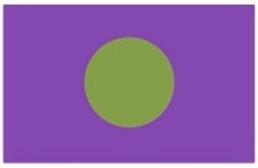

Figure 5-1

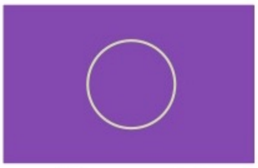

Figure 5-2

In Fig. 5-1, a purple rectangular region has a green circular region inside it. The green region has a boundary, also circular.

In Fig. 5-2, a purple rectangular region has a circular region inside it, but this time, this region is also purple, with a circular boundary.

We may now conceptualise 'circle' in two different ways:

Circle (CONCEPT 1): A circular region

Circle (CONCEPT 2): The boundary of a circular region

Euclid defines *Circle* as follows, which is the same as our CONCEPT 1:

Circle (CONCEPT 1: DEF): A circle is a plane figure bounded by one curved line, and such that all straight lines drawn from a certain point within it to the bounding line, are equal."

Euclid, *Elements*, Book I[1]:4

Given this definition, we may define *Circle* (CONCEPT 2) as follows:

Circle (CONCEPT 2: DEF): A circle is a curved line in which all straight lines drawn from a certain point within it to any point on the circle are equal.

This is a declarative definition that takes a circle to be an object. Instead, if we think of a circle (CONCEPT 2) as a trajectory of a point, we can define CONCEPT 2 as follows:

Circle (CONCEPT 2: DEF Alt): A circle is the trajectory of single point such that all straight lines drawn from a single point to the point in motion are equal.

In modern terminology, CONCEPT 1 is called a 'disk', and CONCEPT 2 is called a 'circle'. (See the Wikipedia entry on 'circle' at:
https://en.wikipedia.org/wiki/Disk_(mathematics))

Whether we are talking about CONCEPT 1 or CONCEPT 2, the point is the 'centre' of the circle, the boundary is called the 'circumference', and the straight line from the center to a point on the circumference is called the 'radius'. A straight line from one point on the circumference to a point on the opposite side, passing through the center, is called the 'diameter'.

Exercise 1

Prove that the length of the diameter of a circle is twice the length of its radius.

Something to think about:

Do circle as a boundary and as a region have length? Do they have areas?

As pointed out in Unit 4, we cannot answer these questions until we define the concepts of *region* and *boundary*. So, following Euclid's strategy, let us postulate our answers as axioms:

Axiom 1: Regions have area.

Axiom 2: Boundaries of regions are lines that have length, but no breadth or area.

We had asked if a point as an intersection between two lines has area. Axiom 2 above expresses our decision to attribute the property of length without breadth. If so, the intersection between two lines has neither breadth nor length.

That means that a circle as a circular line (CONCEPT 2) has length but no breadth; and a circle as a circular region (CONCEPT 1) has area.

Needless to say, we could have chosen to say that a circular line has both length and breadth. That is what we did in §4.2, where we chose to attribute the property of breadth to lines. Here we are taking a different track, and moving back to Euclidean geometry.

Exercise 2

TASK 1: Keeping in mind the Unit 1 discussion of the two ways of defining lines in terms of boundaries, propose a definition for 'circular line', such that:
(a) circle is a sub-category of circular line,
and then propose another definition for 'circular line', such that:
(b) circle is distinct from circular line.

TASK 2: Which of the two concepts in Task 1 would you regard as the better one? State your reasons.

Task 3: Within an axiomatic system which incorporates Axioms 1 and 2, try to come up with definitions of length, breadth, and area.

Instead of defining length or clarifying the concept of length through axioms, Euclid set up the concept of ***congruence*** to replace the notion of 'same length' (or equal length). In this system:

> Two lines A and B are congruent iff one of them can be placed on top of the other such that they coincide exactly.

Suppose we do the following activity in our mind:

> Draw a straight line A, and copy paste it as A'.

You would agree that A and A' would be congruent, and hence will have the same length. Likewise:

Draw a semi-circular line B, and copy paste it as B'.
Draw a wavy line C, and copy paste it as C'.

Lines B and B' will be congruent, and so will lines C and C'. The members of each pair would be have the same length. But what about the comparative lengths of A, B and C? How do we tell if they have the same length, or whether one of them has greater length than another?

We cannot answer that question, because we have not defined or clarified length in such a way that we can compare lengths of lines with distinct shapes.

Exercise 3

Within an axiomatic system which incorporates the definition of length, and Axioms 1 and 2 in §4.2, is it possible to check if two lines that are not congruent have the same length, or if one of them has greater length? Prove your answer.

5.2 Circle Theorems

You must have encountered the so-called circle theorems in school. [Remember theorems like: "A triangle in which one of the sides is the diameter of a circle, and every vertex is on the circle, is a right-angled triangle."] Circle theorems are about the correlation between circles and polygons. Many of them are correlations between circles and triangles. We will state a few of them as conjectures. Try to prove them to establish them as theorems. Resist the temptation to do an Internet search for a proof that someone else has done.

Let us begin with the idea of circumscription and inscription.

Conjecture 1:
For every triangle, there exists exactly one circle that circumscribes it.

Conjecture 2:
For every triangle, there exists exactly one circle that is inscribed in it.

To repeat what we hope is now part of your blood stream, we can neither prove nor disprove these conjectures without defining circumscription and inscription. Let us try these definitions:

Circumscription (DEF):	A circle circumscribes a triangle iff every vertex of the triangle is on the circle.
Inscription (DEF):	A triangle inscribes a circle iff the circle touches every side of the triangle.

These definitions have a conceptual flaw in that they don't let you see what is shared by the two concepts that the words *circumscribe* and *inscribe* refer to. Suppose we use the two words as inverses, and hold that:

A geometric figure A circumscribes a geometric figure B iff B is inscribed in A.

If we accept this view, we do not need two different concepts. We can use the word circumscribe in such a way that Y is inscribed in X means the same as X circumscribes Y. To illustrate, consider these diagrams:

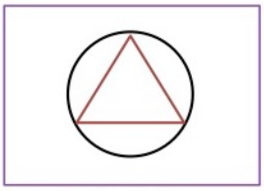

Figure 5-3

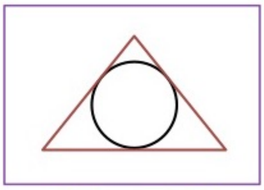

Figure 5-4

Given what we have said about what we want the words *circumscribe* and *inscribe* to mean, we may describe the relation between the circle and the triangle in these two figures as follows:

~ In Fig. 5-3, the circle circumscribes the triangle.
  (= The triangle is inscribed in the circle.)

~ In Fig. 5-4, the triangle circumscribes the circle.
  (= The circle is inscribed in the triangle.)

This conceptualisation of circumscription and inscription deviates from what you may have learnt in school. But here is the way the Wikipedia entry on inscription conceptualises it:

> "In geometry, an inscribed planar shape or solid is one that is enclosed by and "fits snugly" inside another geometric shape or solid. To say that "figure F is inscribed in figure G" means precisely the same thing as "figure G is circumscribed about figure F"." (https://en.wikipedia.org/wiki/Inscribed_figure)

If we adopt the Wiki conceptualisation, how do we define 'circumscribe'?

Exercise 4

TASK 1: Complete the following definition:

Figure X circumscribes figure Y iff …

Using that definition, prove or disprove Conjectures 1 and 2.

TASK 2: Take a look at the eight circle theorems at
http://www.timdevereux.co.uk/maths/geompages/8theorem.php

For your own practice, treat these theorems as conjectures and try to prove as many of them as possible. It is okay to draw upon your memory of the proofs that you are familiar with from school, but you must state them in terms of the structure of premises-derivation-conclusion (PDC) illustrated in the previous Units.

Resist the temptation to do an Internet search to find already available proofs, because that would rob you of a learning opportunity. To develop the capacity to come up with and write proofs with clarity and precision, what matters is the effort that YOU put into this task (either individually or in groups), not whether you are successful in coming up with a proof, or whether an expert judges it to be valid.

5.3 Circles and Regular Polygons

We are now ready to investigate an interesting question:

Can a circle be a regular polygon?

Notice that we are not asking whether a regular polygon can ***approximate*** a circle. We are asking if there are regular polygons that ***are*** circles. There is a world of difference between the two.

A brief introduction to the terminology first.

A polygon is ***equilateral*** iff its sides are of the same length.
A polygon is ***equiangular*** iff its angles are equal.
A polygon is ***regular*** iff it is equilateral and equianglular.

(Before going further, it may be useful to go to the Wikipedia entry on regular polygon at https://en.wikipedia.org/wiki/Regular_polygon and read the introductory paragraph, as well as the material under “General Properties”.)

Now let us do a thought experiment.

Draw a regular polygon in your mind.
Inscribe a circle inside it.
Now circumscribe the polygon with a circle.
Gradually increase the number of sides of the polygon.
As the number of sides increases, you will see this in your mind’s eye:

As the number of sides increases, the distance between the two circles decreases.

To prove that a circle is a regular polygon, you will have to prove that as the number of sides increases, the outer circle and the inner circle coincide, which means that they will also coincide with the polygon.

Can you prove or disprove the claim that it is possible to do so?

[Clue: To construct that proof, use the definitions of ‘polygon’ and ‘circle’.]

Exercise 5

Think about the axioms and definitions we have so far.

In Euclidean Geometry, there is at least one point between any two points, however small the distance between them. This implies that every line, however short, has infinitely many points.

In the non-Euclidean Geometry of §4.2, points can be adjacent, and every finite line has a finitenumber of points.

To prove or refute the conjecture that every circle is a regular polygon;
Did you use a Euclidean system, or a non-Euclidian one?
Would your answer change if you use the other system?

5.4 Degrees of Freedom

For a clearer understanding of the two concepts denoted by the word *circle*, let us try a thought experiment.

Imagine a circular line suspended ten meters above the ground. This line does not form a complete circle; it is only four-fifths of a circle. For a circle to be complete, the line has to return to itself.

Imagine an ant walking along that line. The ant can go backwards and forwards on the line, but cannot step off the line. If she does — walks from one point on the line towards the center of the circle — then she would fall down. In this situation, we say that the ant has only one degree of freedom, namely, backwards and forwards. Even if the circular line is a complete circle, such that if the ant keeps walking it will return to where it started from, the degree of freedom would be the same.

Now imagine a circular disk suspended ten meters above the ground. The ant can now walk not only backwards and forwards along the boundary of the disk, but also backwards and forwards along any diameter line, without falling off the disk. In this situation, we say that the ant has two degrees of freedom.

Now imagine a spherical surface — the surface of a tennis ball. Imagine the ant walking on the surface of the ball. It cannot pierce through the ball to get to the center or to the opposite side. Studying the movement of the ant would be part of the geometry of spherical surfaces which are two-dimensional.

How about if the ball is not a tennis ball but a ball of rice? This time, the ant can burrow through the rice, and get to the center and to the opposite side. Studying the ant's movement here would be part of the geometry of a three-dimensional ball.

How many degrees of freedom does the ant have in these two cases? (Keep going till you feel the smoke coming out of your ears. ☺)

5.5 Summing up

A salient point that emerged in our discussion in Units 4 and 5 is the importance of conceptualising the meaning of an academic term in a theory before we try to define it. This means asking ourselves:

> What is the concept we need, and are looking for?
>
> Having decided on the concept, how do we define it?

This distinction came up quite starkly in our conceptualisation of lines and points in Unit 4 and of circles in Unit 5.

Such conceptual clarification — the process of conceptualising and defining — is important in all domains of inquiry and research. If you wish to assert the existence or non-existence of the soul, it is important to ask what you mean by 'soul' and figure out the meaning sufficiently clearly so that you can assert or deny its existence. Is the concept of 'soul' distinct from the concept of 'mind'? If yes, what is the difference? Does the soul exist after the death of the physical body? If it does, does it retain the mental properties of the person before death? If it does, does it wander the earth, or go to heaven, or hell? Does the soul have supernatural powers to influence the physical world? (Can a soul pick up a stone and hurl it at a window?)

Without reflecting on such questions and arriving at a decision, asserting or denying the existence of souls is meaningless. The next step would be to define the concept with as much clarity and precision as you can muster, such that you can discuss the issue of the existence of the soul with others, present arguments in support of or against its existence, and perhaps engage in a debate on its existence.

Another learning point in Unit 5 was the distinction between

the ***properties of an entity*** (e.g., of polygons, of circles)

and

the ***correlation between the properties*** of two (or more) entities (e.g., between circles and polygons).

A third point was the need to revise and refine the statements we encounter in textbooks or the classroom.

In all of this, you have been learning the art and craft of inventing and defining concepts, and coming up with and articulating conjectures and proofs.