UNIT 4: A THEORY OF LINES AND POINTS

4.1 Lines without Width and Points without Size

When children first learn about the concept of lines and points, their experiential understanding of these concepts is that of the marks we make on a piece of paper, or on a whiteboard. And more often than not, this is the understanding that they carry into adulthood. This is true, regardless of the words and symbols that accompany the concepts.

Within this conception of lines and points, the marks in Fig. 4-1 are ***lines***, and the ones in Fig. 4-2 are ***points***.

Figure 4-1

Figure 4-2

Such textbook diagrams often get in the way of a conceptual understanding of geometry. The reason is simple: when learners are introduced to Euclidean geometry, they are told:

> Lines have no width.
> Points have no width and no length; they have zero size.

For learners (whether children or adults), this makes no sense, because the marks in Fig. 4-1 do have width, however small; and the marks in Fig. 4-2 have some area, no matter how small. The experiential concept contradicts what they are taught. But as they have neither the conceptual clarity nor the vocabulary to voice (or even recognise) their discomfort, especially given the culture of obedience and blind faith that education systems promotes, they have to accept what the textbooks and teachers say.

There is an alternative way of introducing learners to the Euclidean concept of lines and points without creating this dissonance. This would be to define lines and points as follows:

LINE (DEF-1): A line is the boundary (edge) of a region.
(DEF-2): A line is a boundary between two regions.

POINT (DEF): A point is an intersection between two lines.

To elaborate, consider Figs. 4-3 and 4-4.

Figure 4-3

Going by DEF-1 of a line, how many LINES does Fig. 4-3 have? Let us take a look through our mind's eye, backed by our physical eye. The lines are:

1) a. the edge of the purple rectangle;
b. the edge of the orange region; and
c. the edge of the green region.

How about if we go by DEF-2? The lines are:

d. the boundary between the purple region and the white region;
e. the boundary between the purple region and the green region;
f. the boundary between the purple region and the orange region; and
g. the boundary between the green region and the orange region.

Do you see a difference in the consequences of DEF-1 and DEF-2 of lines? Notice that the boundary between the green region and the orange region in Fig. 4-3 is a line by DEF-2, but not by DEF-1. This is because, unlike the other lines which are boundaries of regions, and are closed figures, the 'line' between the the green region and the orange region is not a closed figure, nor is it the boundary of any region.

Let us now consider Fig. 4-4.

Figure 4-4

As in the case of Fig. 4-3, the perimeter of the rectangle is the boundary between the purple region of the rectangle and the white region outside it, and is a line by DEF-2.

We also have a white line and an orange line in the figure. Each of them counts as a line by DEF-2, because they form the boundary between two regions of the purple rectangle. They are not lines by DEF-1, however, since they do not form the boundary of a region.

A note to remember: That there are coloured regions in Figs. 4-3 and 4-4 is irrelevant to the concepts we are trying to clarify for ourselves. Colour is not a concept in the world of geometry.

The main point about the above discussion of the concept of line, which we all take for granted as something we know, is to emphasise that

- ~ we cannot talk about a concept in an academic discussion without defining that concept;
- ~ a concept can be defined in more than one ways;
- ~ different definitions may have distinct logical consequences; and
- ~ the definition that we choose as the most appropriate depends on those consequences, and how they contribute to developing a coherent theory.

Given our definitions of a line, are there any lines in Fig. 4-1? Yes, the boundary of the rectangle is a line. But what about the marks inside the rectangle? Given our definition, these marks are not lines, because they do not mark the boundary of a region, nor the boundary between two regions.

Let us turn to points for a moment. Given our definition of a point as an intersection between two lines, the intersection between the white line and the orange line in Fig. 4-4 counts as a point. Are there any other points in Fig. 4-4? No, because there are no other intersections between two lines in the figure.

Given our definition of a point, are there any points in Fig. 4-2? No, because there are no intersections between the two lines in the figure.

Do the lines in Figs. 4-1, 4-3 and 4-4 have width? We cannot answer that question, because we don't know what the word *width* means in the logically possible imagined world we have just created, the world populated by regions, boundaries, lines, and intersection between lines.

Does the point in Fig.4-4 have length? As in the case of *width*, we cannot answer that question unless we define the concept denoted by the word *length*. Again, we are in a logically possible imagined world apprehended only through our mind's eye, not the world of our sensory experience apprehended through our physical eye, ear, nose, tongue, and skin. .

For exactly the same reason, we cannot answer questions about the *area* or *volume* of an object in the world that we have constructed so far, at least not until we have defined the concepts denoted by the words *length*, *width*, *area*, and *volume* with sufficient clarity and precision.

Our definitions of *line* and *point* use the terms *region*, *edge*, and *boundary*. Let us ask, do the *regions* in Figs. 4-1 to 4-4 have areas? We don't know, because though we have used the words *region*, *edge*, and *boundary* in our definitions, we have not defined the concepts that the words refer to. So for now, we must take region, edge, and boundary as ***undefined concepts*** (also called ***primitive concepts***) in the theory of lines and points we have just created.

4.2 Lines with Width and Points with Area

Do lines have width? Do points and lines have area? In §4.1, we took the position that we cannot answer these questions because we have not defined width and area.

The concepts of length and width are not defined in Euclid's *Elements* either. As a result, the concept of area also remains undefined. But, aswe saw earlier, Euclid axiomatically adopted 'no' as the answer to the above questions, and postulated that lines have no width; and points have no length or width, and hence no area.

But what if we take the opposite position? Let us try, by first defining the length of lines:

> LENGTH (DEF): The length of a line is the number of points it contains.

This means that if a finite line A has more points than line B, A is longer than B. But in Euclid, every finite line has infinitely many points. So the statement that A has more points than B does not make sense.

What if we set up the following axiom to solve this problem:

> **Axiom:** Every line of finite length has a finite number of points.

In Euclid, because points have zero length and zero width, adding points together does not create a line with non-zero length. Furthermore, no matter how close two points are, there is always at least one point between them. Hence, no two points are adjacent, that is, next to each other.

In the world of integers, numbers 3 and 4 are adjacent because there is no integer between them. So are integers 458 and 459: there is no integer between them. In contrast, 3 and 7 are not adjacent, nor are 458 and 464.

In the world of rational numbers, however 3 and 4 are not adjacent. For instance, 3.6 lies between 3 and 4. How about 3 and 3.1? Are they adjacent? No, because 3.13 lies between them. What about 3.13 and 3.14? They are not adjacent either because 3.136 lies between them.

So, points in Euclidean geometry are like rational numbers rather than like integers, because they cannot be adjacent. In the geometry that we are setting up now, we are treating points like integers, where two points can indeed be adjacent.

We now have two theories of geometry: the rational number geometry of Euclid's *Elements*, and the integer geometry that we are considering. Given this situation, it is legitimate to ask: Do the two theories constitute two distinct theories of geometry? Or are they not distinct? Do they yield distinct theorems, which are logically contradictory?

Let us take a look.

Take the concept of bisection in Euclidean geometry. Theorem 10 in Euclid says:

Every finite line is bisectable.

This means that every finite line can be cut into two lines which are congruent (= of the same length).

Is this a theorem in a rational number theory of geometry? How about in an integer theory of geometry?

Exercise 1

Task: Define BISECTION. Then figure out if the following conjecture is true:

There exist lines which are not bisectable.

If you think it is true, prove it.

If you are given two theories of the same thing, and asked if they constitute two distinct theories, or the same theory, engaging with Ex. 1 will help you think towards an answer to that question.

4.3 Parallel Lines

Intuitively, we judge the lines in Fig. 4-5 to be "parallel", and those in Fig. 4-6 to be not parallel.

Figure 4-5 ***Figure 4-6***

Underlying this intuition are certain properties we attribute to parallel lines, all of which are satisfied by Fig. 4-5, but not Fig. 4-6:

Table 1

	Property	Fig. 4-5	Fig. 4-6
a.	Equi-distance: The distance between the two lines is invariant even when extended indefinitely on both sides,	**YES**	**NO**
b.	Non-intersectability: The lines never intersect even when extended indefinitely on both sides.	**YES**	**NO**
c.	Perpendicular intersection: An intersecting straight line perpendicular to both the lines is possible.	**YES**	**NO**
d.	Conservation of the sum of internal angles: If a straight line intersects the two lines, the sum of internal angles on the same side is two right angles.	**YES**	**NO**

In Euclidean geometry, the properties in Table 1 converge: any pair of straight lines that has one of these properties also has the others. Hence, we may take any of these properties as definitional of the concept of parallel, and derive the remaining properties as theorems.

In non-Euclidean geometries, it is possible for a pair of lines to have one or more of these properties but not the others.

The term "parallel" is conventionally used only in the context of straight lines. However, unless the definition specifies that the term is restricted to straight lines, we can take them as holding on curved lines as well. Consider, for instance, the following pairs of curved lines.

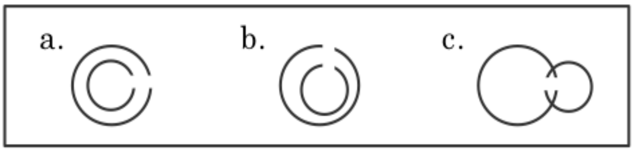

Figure 4-7

The pairs of lines in Fig. 4-7 are all circular (and therefore curved).

In Fig. 4-7a, the lines are concentric. Therefore:

- they are equidistant;
- they cannot intersect;
- a straight line drawn through the center would be perpendicular to both lines (provided we define *perpendicularity* and angle in a way that they apply to curved lines; and
- given any straight line intersecting the two lines, the sum of the internal angles on the same side is two right angle.

Thus, Fig. 4-7a exhibits all the properties in Table 1. Unless the definition restricts the concept of parallel to straight lines, the lines in Fig. 4-7a would count as parallel.

How about Figs. 4-7b and 4-7c? Using what is given in Table 2, work out how these two figures relate to the properties of parallelness.

Table 2

	Property	**Fig. 4-7a**	**Fig. 4-7b**	**Fig. 4-7c**
a.	Equi-distance	**YES**	**NO**	**NO**
b.	Non-intersectability	**YES**	**YES**	**NO**
c.	Perpendicular intersection	**YES**	**NO**	**NO**
d.	Conservation of sum of internal angles	**YES**	**NO**	**NO**

Whether or not the lines in Fig. 4-7b are parallel depends on how we define the term. If we take (b) in Table 2 as its defining property, then the lines in the figure are parallel, but not otherwise.

How about the lines in Fig. 4-7c? They are not parallel whichever property we choose as definitional of the concept parallel.

Exercise 2

Consider the following definitions of parallel lines:

Definition 1: Two lines A and B are parallel if and only if there exists a line C perpendicular to both.

Definition 2: Two lines A and B are parallel if and only if they are equidistant.

Draw figures to demonstrate that both these definitions are flawed. Then revise the definitions to remove the flaws.

In our discussion of parallel lines, the concepts of angle and of straight vs. curved lines turned out to be crucial. We are going to explore curved lines further in Unit 5. But in the meantime, it may be a good idea for you to invest some time to define straight line, angle, and vertex, and discuss your ideas with whoever is interested.

4.6 Summing up

In Unit 1–4, we used the methodological strategies of axiomatic inquiry in theory construction. An axiomatic system consists of premises (axioms and definitions), derivations, and conclusions. And the truth of a conclusion is proved by deducing it from the premises.

Mathematical theories are axiomatic systems. Theories outside of mathematics also have an axiomatic component. But in addition, they have another component, in which the predictions of the theory are matched against something outside the axiomatic component. Scientific theories, for example, also include an observational component. They are subject to the norm that the predictions (logical consequences derived from the premises) of the axiomatic component of the theory must fit with the observational generalisations (what is observationally established). Given that mathematical theories are about logically possible imagined worlds that we create, this norm does not hold on them.

Two important properties of the axiomatic mode of inquiry that we found in this Unit are:

A. In axiomatic inquiry, a question about something cannot be answered unless that something is defined clearly and precisely.

B. If we change an axiom or a definition, the truth of the conclusions may also change.

Finally, given the kind of logic used in mathematics, if we add a new axiom or definition to the system, what was established previously as true remains true. As we will see later, this is not the case with the axiomatic systems of scientific theories.