UNIT 3: A THEORY OF POLYGONS

3.1 Looking Back

In Units 1 and 2, we drew attention to some of the ***methodological strategies*** of theory construction:

defining, categorising, abstracting, integrating, and ***proving***.

These are relevant in all domains of academic research. You could think of these strategies also as *techniques*, or *tools* used in builiding knowledge. The words we use for these strategies are in their verb form. Corresponding to them are also their noun forms, which we think of as ***concepts*** of academic knowledge and inquiry/research:

definitions, categories, abstractions, integration, and ***proofs***.

In Unit 2, we unpacked the concept of proof in terms of a structure that involves the concepts of ***premises, derivation,*** and ***conclusion*** (PDC structure of proofs). We made a distinction between two kinds of premises: ***axioms*** and ***definitions***. We saw the methodological strategy of ***axiomatising*** — of explicitly stating the axioms and definitions of a proof, such that we can critically evaluate the ***validity*** of proofs and the ***credibility*** of theories. The concept corresponding to axiomatising, we said, is that of an ***axiomatic system***.

These strategies and concepts were introduced in the specific context of geometry (and biology). However, they play an important role in theory construction in all domains of research, ranging from mathematics, the physical-biological-human sciences, and the humanities. We will therefore refer to them as ***transdisciplinary*** strategies and concepts, *trans-* in the sense that they cut across specialised fields, disciplines, and discipline groups, and exist at a more abstract level. To take an example of what we mean by transdisciplinary, the so-called *theory of classical mechanics* is specific to the highly specialised field that studies gravity and motion within the discipline called physics; but the concept of ***theory*** is a trans-disciplinary concept. It is in this sense that we talk about transdisciplinary concepts.

Take another example. The concepts of gravity, velocity, and acceleration are specific to physics, and the concept of skeletal structure, and of cells being composed of biomolecules is specific to biology. But the concepts of ***structure***, ***compositionality***, and very concept of ***concept*** are transdisciplinary.

In §1.5.3, we made a distinction between axioms that are specific to a particular field or discipline, and ***general axioms*** that are part of all academic knowledge. General axioms are transdisciplinary. We may make the same distinction for definitions. The definition of force as *that which causes a change in the velocity of an inanimate entity* is specific to physics. But the definition of force as *that which causes change* is a transdisciplinary definition, extendable to the animate, even human domains.

The study of knowledge includes the nature of knowledge, ways of arriving at it, critically evaluating it, and proving claims to establish them as part of knowledge. This study is called ***epistemology*** in philosophy, and ***cognitive science*** in the sciences. (Cognising means knowing; and *cognise* and *know* both derive from the Indo-European root *gno*-.)

The kind of epistemology discussed in philosophy textbooks and in philosophy classrooms does not cover the epistemology of ***academic knowledge***. It does not engage with questions such as:

What is the distinction between mathematical proofs and scientific proofs?

What is the distinction between correlational theories and causal theories?

Given a question, how do we decide what kind of reasoning to use?

In this book, what we are interested in is the *epistemology of academic knowledge* that functions as the foundations for research in all domains of academic knowledge.

In what follows, we will take a closer look at the *concept of proof*, and the role of *categorisation and subcategorization* as an important methodological strategy for *integrating special theories into a general theory*.

3.2 The Concept of Proof

In Unit 2, we introduced the concept of proof in terms of a three-part structure: premises, derivation, and conclusion (PDC). Let us go through a few examples to get a firmer grip on this concept.

Conjecture to be proved: Athena was born in 1903.

Premises

P1. If Plato has a beard, then Aristotle dislikes mangoes.
P2. If Aristotle dislikes mangoes, then Athena was born in 1903.
P3. Plato has a beard.

Derivation: Steps

S1:	If Plato has a beard, then Aristotle dislikes mangoes.	(P1)
S2:	Plato has a beard.	(P3)
S3:	Therefore, Aristotle dislikes mangoes.	(by S1, S2)
S4:	If Aristotle dislikes mangoes, then Athena was born in 1903.	(P2)
S5:	Therefore, Athena was born in 1903.	(by S3, P2)

Conclusion

Athena was born in 1903. (QED)

NOTE: QED is an abbreviation for the Latin expression Quod Erat Demonstrandum, which means ‘that which is to be demonstrated (= to be proved.)

Let us take another example:

Conjecture to be proved: Athena is taller than Apollo.

Premises

P1. Athena is taller than Xena.
P2. Xena is taller than Plato.
P3. Plato is taller than Apollo.

Derivation: Steps

S1:	Athena is taller than Xena.	(P1)
S2:	Xena is taller than Plato.	(P2)
S3:	Therefore, Athena is taller than Plato.	(by S1, S2)
S4:	Plato is taller than Apollo.	(P3)
S5:	Athena is taller than Plato.	(S3)
S6:	Therefore, Athena is taller than Apollo.	(by S4, S5)

Conclusion

Athena is taller than Apollo. (QED)

In Unit 1 (6) [read as: Unit 1, Item (6)), we gave an example of a derivation of the conclusion that Right-Angled Triangles have three angles. Given below is roughly the same proof, but stated in terms of premises, derivation, and conclusion.

Conjecture to be proved: RATs have three angles.

Premises

P1. A Triangle has three angles.
P2. An RAT is subcategory of Triangle.
P3. The properties of a category are inherited by their subcategories.

Derivation: Steps

S1: A Triangle has three angles. (P1)
S2: An RAT is subcategory of Triangles. (P2)
S3: The properties of a category are inherited by its subcategories. (P3)
S4: Therefore, RATs have three angles. (by S1-3)

Conclusion

RATs have three angles. (QED)

The discipline-specific axiom of subcategorization in (P2) and the transdisciplinary axiom of the logical inheritance of properties in (P3) are central to this proof. Proofs that appeal to subcategorization and the accompanying inheritance of properties are essential for integrating the theories of RATs, ETs, and Triangles into a single theory. This point can be generalised as:

> Categorisation and subcategorization serve an important function in the integration of academic knowledge.

We will have more occasions to use this strategy in the integration of other theories.

In school, you must have come across Euclid's Proof of the infinity of prime numbers. If you haven't, please do an Internet search, and take a look at two or three versions of the proof. The following YouTube video is fairly easy to understand:

"Euclid's Proof that there an Infinite Number of Prime Numbers"
at https://www.youtube.com/watch?v=OxGRl0phiB4

3.3 Properties and Relations

In §1.6, we distinguished between ***descriptions*** and ***theories***. A description of an entity is a body of information about that entity. In disciplines like chemistry, the description of an element is a set of structural or behavioural properties of that element. (e.g., copper is malleable but glass is brittle; petrol is inflammable, water is not; and so on.) A theory takes a number of description fragments, generalises them, and then configures the generalisations into a logical structure of premises-derivation-conclusion. A theory is subject to the requirement that the smallest possible number of premises should yield the widest range of conclusions.

When we are looking at the properties of entities that we want our theory to predict (to derive as a conclusion or a theorem), it is useful to think of not only ***properties***, but also ***relations***. Here are a few examples to illustrate the difference:

Property	*Relation*
Zeno IS OLD.	Zeno IS OLDER THAN Plato.
Zeno IS YOUNG.	Zeno IS YOUNGER THAN Plato.
Zeno IS MARRIED.	Zeno IS MARRIED TO Athena.
Zeno IS A TEACHER.	Zeno TEACHES Plato.

As in the case of axioms and definitions, properties and relations can be of a specific field, a discipline, a discipline group, or transdisciplinary. We are interested in both discipline-specific as well as transdisciplinary properties and relations.

Transdisciplinary relations that appear in theories across disciplines include:

Subcategorization: *x* is A SUBCATEGORY OF *y*.

Compositionality: *x* is COMPOSED OF *y, z, …* (Variants: is MADE UP OF / is A CONSTITUENT OF, is DECOMPOSABLE INTO, …)

Ordering: *x* is ORDERED PRIOR TO *y*. (Variants: is RANKED HIGHER THAN; PRECEDES)

Logical consequence: *x* is a LOGICAL CONSEQUENCE OF *y*.

Logical contradiction: *x* LOGICALLY CONTRADICTS *y*.

Equality: *x* IS EQUAL TO *y*. (Variant: IS EQUIVALENT TO, IS AN ANALOGUE OF, IS A HOMOLOGUE OF, …)

Correlation: *x* CORRELATES WITH *y*.

Causation: *x* CAUSES *y*

Instantiation: *x* is AN INSTANCE OF *y*. (Variants: is A MEMBER OF set/category y, is AN EXAMPLE OF y, is A SAMPLE OF y, …)

Negation: x is THE NEGATION OF *y*. (Variant: is THE OPPOSITE OF.) (Also see logical contradiction)

We have already discussed subcategorization (§1.3, §1.4, §2.2, §2.3). In conjunction with the axiom of the logical inheritance of properties (Unit 1 (5)), the use of the subcategory relation allows us deduce otherwise arbitrary properties of a category from its mother category, thereby facilitating the integration of special theories into general theories.

We have seen the appearance of compositionality in geometry in statements like:

A triangle *is composed of* exactly three vertices, and exactly three straight lines connecting them.

A quadrilateral *is composed of* exactly four vertices, and exactly four straight lines connecting them.

A pentagon i*s composed of* exactly five vertices, and exactly five straight lines connecting them.

Compositionality, then, is a ***part-whole relation.*** We say:

A human body is composed of organs;
An organ is composed of tissues;
A tissue is composed of cells;
A cell is composed of molecules;
A molecule is composed of atoms;
An atom is composed of particles.

In the statements of the form "*x is composed of y, z…* ," *x* is the whole, and the rest are parts of that whole.

As in geometry (and in biology), it would be useful to explore compositionality in domains like arithmetic — for instance, we may view the operations of addition, subtraction, multiplication and division as involving compositionality.

The idea of compositionality in arithmetic calls for a short explication. The use of the term operation implies the idea of an input, an action performed on the input (or a process that the input undergoes), and the resultant output:

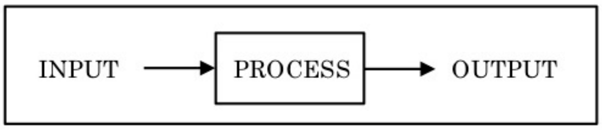

Figure 3.1

This way of thinking of input-process-output based on the metaphor of ACTION and CHANGE is built into the term *operation*. It is also characteristic of the concept of algorithm in computer science, where an algorithm is a set of explicit procedures to do something.

When we say:

10 is the SUM OF 6 and 4, or
10 is a MULTIPLE OF 5,10 is the PRODUCT OF 2 and 5,
10 is DIVISIBLE BY 5,

we may indeed think of these concepts in terms of the input-output metaphor. On the other hand, we may also think of them as binary relations. In this view, "X is the sum of Y+Z" is a binary relation between X on the one hand, and Y and Z on the other. In our example, X is 10, Y is 6 and Z is 4. 10 is divisible by 5 is a relation between 10 and 5. We would like to suggest that the transdisciplinary core of these operations/relations is the concept of compositionality.

It would be useful to pause and reflect on where these relations appear, in all the disciplines that you have encountered in your education/research.

3.4 Towards an Integrated Theory of Polygons

In Units 1 and 2, we introduced the idea of categorising and subcategorising as a methodological tool to integrate two or more special theories into a single general theory. For example, we integrated three otherwise unrelated theories — of RATs, ETs and Triangles — into a single theory of Triangles by postulating that RATs and ETs are subcategories of Triangles (§1.3, §1.4).

Similarly, we integrate the otherwise unrelated theories of Squares, Rectangles, Rhombuses, and Parallelograms into a single theory of Quadrilaterals, by postulating the following axioms:

Squares are a subcategory of Rectangles.
Rectangles are a subcategory of Parallelograms.
Squares are a subcategory of Rhombuses.
Rhombuses are a subcategory of Parallelograms.
Parallelograms are a subcategory of Quadrilaterals.

To this, we may add:

Triangles, Quadrilaterals, Pentagons, Hexagons,…
are subcategories of Polygons.

3.5 Categorization and Compositionality in Conjectures

Let us try some mental exercises. Imagine a Square, by constructing a Square in your mind. In that Square, draw diagonals. How many diagonals can you draw? No more than two, right?

Now imagine a Rectangle, and draw diagonals in it. How many can you draw? Again, no more than two, right? Do the same thought experiment with a Rhombus. How many diagonals? No more than two?

Let us generalise, to a Parallelogram, and to any Quadrilateral.

Based on the sample of pictures in your mind, can you come up with a conjecture on diagonals? To kick off the task, let us begin with one that is maximally general and obviously false, and one that is most specific that we feel is obviously true.

Conjecture 1: A geometric figure can have two but no more than two diagonals.

Conjecture 2: A square can have two but no more than two diagonals.

Exercise 1:

TASK 1: Try to prove (or disprove) Conjectures 1 and 2. To do this, you need to first define the concept of diagonal.

You also need to separate the two strands in the conjecture: showing that:

(i) there exist geometric figures/squares that can have two diagonals;
(ii) no geometric figure/square can have more than two diagonals.

TASK 2: Ask yourself if there are geometric figures with three or more diagonals. For this, it would be useful to do a thought experiment of constructing geometric figures with three or more diagonals.

Complete these tasks before you proceed to the next exercise. Otherwise you would miss out on an important opportunity for experiential learning through a minds-on activity (as opposed to a hands-on one).

Exercise 2

TASK 1: Using what you have learnt in this Unit, especially by going through Ex. 1, come up with conjectures on the categories of pentagons, hexagons, and septagons.

Formulate a conjecture on diagonals in polygons.

TASK 2: Now try to formulate a conjecture on diagonals in geometric figures in general. Does this instruction trigger an unease in your mind? Why? Think of a possible reason, and state it in writing, as clearly and precisely as you can.

Let us go back to the geometry that we all learnt before we got to Class 10. We were exposed to the terminology of Right-Angled Triangle, Equilateral Triangle, EquiAngular Triangle, Isosceles Triangle, Square, Rectangle, Parallelogram, Rhombus, Quadrilateral, Pentagon, Hexagon, … Polygon in school. We were also exposed to the terminology of Point, Straight Line, Curved Line, Vertex, Side, and Diagonal. The tasks in Exs. 1 and 2 are meant to trigger in you an experience of what it is like to formulate conjectures, and prove them.

When we do the thought experiment of drawing a diagonal in a square in our mind, we see in our mind's eye, without having to draw it on a piece of paper, that the square is composed of two triangles. Again, without drawing it on paper, we can see that the two triangles are equilateral isosceles triangles.

Let us write down what we have discovered through our thought experiments:

Conjecture 3: Any square can be divided into two right-angled isosceles triangles.

Conjecture 4: Any two right-angled isosceles triangles (of the same size) can be joined to form a square.

Exercise 3

Conjectures 3 and 4 are stated using the idea of a process of dividing or combining: the statements use the metaphor of CHANGE. They use a 'procedural' language.

TASK: Can you restate the conjectures without implying a process of dividing or combining? Such a statement would use the metaphor of an object, and the relation of compositionality. It needs the use of a 'declarative' language.

This translation from procedural language to declarative language is not easy. but it is worth putting effort into, because it will help you train your intuitions, sharpen insight, and develop the capacity to make connections.

The statements that you arrived at in Exs. 1–3 involve the relations between categories, as well as the relation of compositionality. The combination of these two kinds of relations is integral to the concept of ***structure***. This concept is of value not only in geometry, but perhaps in all domains of academic inquiry. In other words, structure as conceptualised above is a transdisciplinary concept. You could view what we are doing in these exercises as beginning to construct ***a theory of the structure*** of polygons.

We hope that you are tickled by — and not confused by — the title, *Structure of Theories* in Unit 1, and the mention of *Theory of Structure* in the preceding paragraph.

Now, the truth of the propositions in the theory you have constructed in Exs. 1–3 may be 'obvious' to you. But we must remind you that it is based on a sample of one square-triangle pair. You cannot do thought experiments on every square-triangle pair in an infinite population of squares and triangles. So, even though they may be 'obvious' to you, you have not yet established them as true propositions. These propositions are conjectures, they are not yet theorems.

There is another reason why these propositions are only *plausible conjectures*, not theorems. Their truth may be obvious to you, but what if they are not obvious to the others in your research community? What if they demand proofs before they accept the propositions as true?

Exercise 4

TASK: Come up with proofs of the conjectures in your theory of squares and triangles. Write down your proofs with as much clarity, precision, and rigour as you can muster.

We could think of the outcome of Ex. 4 as an example of *simulated research*. It is research in the sense that, to do these exercises, you went through the process of inquiry that is needed in research. And it is *simulated* because the theory that you have come up with is already known to the community of mathematicians, so it is not a contribution to the field. (To qualify as research, the outcome of the process must make a contribution to the existing body of knowledge.)

That it is simulated research rather than actual research should not bother you. If you continue along the journey in this book, and practice several times what you have learnt, taking new research questions each time, you will hone your abilities and be ready to make contributions to the existing body of knowledge in your chosen field of study.

Exercise 5

The theory that we have constructed so far is on triangles and squares.

TASK: Generalise it to construct a theory of triangles and quadrilaterals.

Exercise 6

Triangles and quadrilaterals are polygons.

TASK: Construct an integrated theory of polygons, and present it as a written document, with as much clarity, precision, and rigour that you can muster, as an individual or joint project.

Our practice of the methodological strategies for theory construction in Exs. 1–6 have been restricted to the terrain of geometry. For further practice, and for you to get a first-hand experience of seeing what is transdisciplinary about the methodological tools we have used, let us move from geometry to biology.

Exercise 7

TASK: Based on the biology exercises (Exs. 2–5) in Unit 2, construct an integrated theory of the structure of the taxa and of organs discussed in the exercises, and present your theory as a research article.

This is not an exercise that can be done in a few minutes or a few hours. It is a project that could take you a month or two, perhaps even more. You can do it either as an individual project, or a joint project with one or more collaborators.

3.6 Summing up

In Unit 1, we gave a demonstration of how we can construct an integrated theory of triangles, and in Unit 2, showed how we can construct an integrated theory of quadrilaterals. In §3.1–§3.4, we demonstrated how we can construct an integrated theory of polygons.

The transdisciplinary tools and concepts that we have used so far for constructing and evaluating theories in the terrain of geometry include:

structure: categorization and subcategorization, compositionality
generalization, integration
premises (axioms and definitions), derivation and conclusion
conjectures, and theorems/predictions; proof

We have also used these tools and concepts for constructing and evaluating theories in biology, in order to show their transdisciplinary nature.